

DISCRETIZING MALLIAVIN CALCULUS

CHRISTIAN BENDER AND PETER PARCZEWSKI

ABSTRACT. Suppose B is a Brownian motion and B^n is an approximating sequence of rescaled random walks on the same probability space converging to B pointwise in probability. We provide necessary and sufficient conditions for weak and strong L^2 -convergence of a discretized Malliavin derivative, a discrete Skorokhod integral, and discrete analogues of the Clark-Ocone derivative to their continuous counterparts. Moreover, given a sequence (X^n) of random variables which admit a chaos decomposition in terms of discrete multiple Wiener integrals with respect to B^n , we derive necessary and sufficient conditions for strong L^2 -convergence to a $\sigma(B)$ -measurable random variable X via convergence of the discrete chaos coefficients of X^n to the continuous chaos coefficients of X . In the special case of binary noise, our results support the known formal analogies between Malliavin calculus on the Wiener space and Malliavin calculus on the Bernoulli space by rigorous L^2 -convergence results.

1. INTRODUCTION

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on a probability space (Ω, \mathcal{F}, P) , where the σ -field \mathcal{F} is generated by the Brownian motion and completed by null sets. Suppose ξ is a square-integrable random variable with zero expectation and variance one. As a discrete counterpart of B we consider, for every $n \in \mathbb{N} = \{1, 2, \dots\}$, a random walk approximation

$$B_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n, \quad t \geq 0,$$

where $(\xi_i^n)_{i \in \mathbb{N}}$ is a sequence of independent random variables which have the same distribution as ξ . We assume that the approximating sequence B^n converges to B pointwise in probability, i.e.

$$\forall t \geq 0 : \quad \lim_{n \rightarrow \infty} B_t^n = B_t \text{ in probability.} \quad (1)$$

The aim of the paper is to provide L^2 -approximation results for some basic operators of Malliavin calculus with respect to the Brownian motion B such as the chaos decomposition, the Malliavin derivative, and the Skorokhod integral by appropriate sequences of approximating operators based on the discrete time noise $(\xi_i^n)_{i \in \mathbb{N}}$. It turns out that in all our approximation results, the limits do not depend on the distribution of the discrete time noise, hence our results can be regarded as some kind of invariance principle for Malliavin calculus.

We briefly discuss our main convergence results in a slightly informal way:

- (1) *Chaos decomposition:* The heuristic idea behind the chaos decomposition in terms of multiple Wiener integrals is to project a random variable $X \in L^2(\Omega, \mathcal{F}, P)$ on products of the white noise $\dot{B}_{t_1} \cdots \dot{B}_{t_k}$. This idea can be made rigorous with respect to the discrete noise $(\xi_i^n)_{i \in \mathbb{N}}$ by considering the discrete time functions

$$f_X^{n,k}(i_1, \dots, i_k) = \frac{n^{k/2}}{k!} \mathbb{E} \left[X \prod_{j=1}^k \xi_{i_j}^n \right]$$

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for pairwise distinct $(i_1, \dots, i_k) \in \mathbb{N}^k$. Our results show that, after a natural embedding as step functions into continuous time, the sequence $(f_X^{n,k})_{n \in \mathbb{N}}$ converges strongly in $L^2([0, \infty)^k)$ to the k th chaos coefficient of X , for every $k \in \mathbb{N}$ (Example 35). This is a simple consequence of a general Wiener chaos limit theorem (Theorem 29), which provides equivalent conditions for the strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of a sequence of random variables $(X^n)_{n \in \mathbb{N}}$ (with each X^n admitting a chaos decomposition via multiple Wiener integrals with respect to the discrete time noise $(\xi_i^n)_{i \in \mathbb{N}}$) in terms of the chaos coefficient functions. As a corollary, this Wiener chaos limit theorem lifts a classical result by [Surgailis (1982)] on convergence in distribution of discrete multiple Wiener integrals to strong $L^2(\Omega, \mathcal{F}, P)$ -convergence (in our setting, i.e. when the limiting multiple Wiener integral is driven by a Brownian motion).

- (2) *Malliavin derivative*: With our weak moment assumptions on the discrete time noise, we cannot define a discrete Malliavin derivative in terms of a polynomial chaos as in the survey paper by [Gzyl (2006)] and the references therein. Instead we introduce the discretized Malliavin derivative at time $j \in \mathbb{N}$ with respect to the noise $(\xi_i^n)_{i \in \mathbb{N}}$ by

$$D_j^n X = \sqrt{n} \mathbb{E}[\xi_j^n X | (\xi_i^n)_{i \in \mathbb{N} \setminus \{j\}}],$$

which is the gradient of the best approximation in $L^2(\Omega, \mathcal{F}, P)$ of X as a linear function in ξ_j^n with $\sigma(\xi_i^n, i \in \mathbb{N} \setminus \{j\})$ -measurable coefficients. Theorem 13 below implies that, if (X^n) converges weakly in $L^2(\Omega, \mathcal{F}, P)$ to X and the sequence of discretized Malliavin derivatives $(D_{[n, \cdot]}^n X^n)_{n \in \mathbb{N}}$ converges weakly in $L^2(\Omega \times [0, \infty))$, then X belongs to the domain of the continuous Malliavin derivative and the continuous Malliavin derivative appears as the weak $L^2(\Omega \times [0, \infty))$ -limit. As the Malliavin derivative is a closed, but discontinuous operator, this is the best type of approximation result which can be expected when discretizing the Malliavin derivative. Sufficient conditions for the strong convergence of a sequence of discretized Malliavin derivatives, which can be checked in terms of the discrete-time approximations, are presented in Theorems 17 and 36.

- (3) *Skorokhod integral*: Defining the discrete Skorokhod integral as the adjoint operator to the discretized Malliavin derivative leads to

$$\delta^n(Z^n) := \lim_{M \rightarrow \infty} \sum_{i=1}^M \mathbb{E}[Z_i^n | (\xi_j^n)_{j \in \{1, \dots, M\} \setminus \{i\}}] \frac{\xi_i^n}{\sqrt{n}},$$

for a suitable class of discrete time processes Z^n , which is in line with the Riemann-sum approximation for Skorokhod integrals in terms of the driving Brownian motion in [Nualart and Pardoux (1988)]. Analogous results for the ‘closedness across the discretization levels’ as in the case of the discretized Malliavin derivative and sufficient conditions for strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of a sequence of discrete Skorokhod integrals are provided in Theorems 9, 19 and 37. When restricted to predictable integrands, the convergence results for the Skorokhod integral give rise to necessary and sufficient conditions for strong and weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of a sequence of discrete Itô integrals (Theorem 21). This result can be applied to study different discretization schemes for the generalized Clark-Ocone derivative (which provides the integrand in the predictable representation of a square-integrable random variable as Itô integral with respect to the Brownian motion B). In this respect, Theorems 24 and 26 below complement related results in the literature such as [Briand et al. (2002), Leão and Ohashi (2013)] and the references therein.

We note that related classical semimartingale limit theorems for stochastic integrals (with adapted integrands) [Jakubowski et al. (1989), Kurtz and Protter (1991)] and for multiple Wiener integrals [Surgailis (1982), Avram and Taquq (1986), Avram (1988)], or robustness results for martingale representations [Jacod et al. (2000), Briand et al. (2002)] are usually obtained in the framework of (or using techniques of) convergence in distribution (on the Skorokhod space).

In contrast, we exploit that strong and weak convergence in $L^2(\Omega, \mathcal{F}, P)$ can be characterized in terms of the S -transform, which is an important tool in white noise analysis, see e.g. [Kuo (1996), Janson (1997), Holden et al. (2010)], and corresponds to taking expectation under suitable changes of measure. We introduce a discrete version of the S -transform in terms of the noise $(\xi_i^n)_{i \in \mathbb{N}}$ and show that strong and weak $L^2(\Omega, \mathcal{F}, P)$ -convergence can be equivalently expressed via convergence of the discrete S -transform to the continuous S -transform (Theorem 1). With this observation at hand, all our convergence results can be obtained in a surprisingly simple way by computing suitable $L^2(\Omega, \sigma(\xi_i^n)_{i \in \mathbb{N}}, P)$ -inner products and their limits as n tends to infinity. However, all these results can be seen as strong and weak invariance principles for Malliavin calculus.

The paper is organized as follows: In Section 2, we introduce the discrete S -transform and discuss the connections between weak (and strong) $L^2(\Omega, \mathcal{F}, P)$ -convergence and the convergence of the discrete S -transform to the continuous one. Equivalent conditions for the weak L^2 -convergence of sequences of discretized Malliavin derivatives and discrete Skorokhod integrals to their continuous counterparts are derived in Section 3. Combining these weak L^2 -convergence results with the duality between discrete Skorokhod integral and discretized Malliavin derivative, we also identify sufficient conditions for the strong L^2 -convergence which can be checked solely in terms of the discrete time approximations. We are not aware of any such convergence results for general discrete time noise distributions in the literature. In Section 4, we specialize to the nonanticipating case and prove limit theorems for discrete Itô integrals and discretized Clark-Ocone derivatives. The strong L^2 -Wiener chaos limit theorem is presented in Section 5, and is applied in order to provide equivalent conditions for the strong L^2 -convergence of sequences of discretized Malliavin derivatives and discrete Skorokhod integrals in terms of tail conditions of the discrete chaos coefficients in Section 6. Finally, in Section 7, we consider the special case of binary noise, in which discrete Malliavin calculus is very well studied, see e.g. the monograph by [Privault (2009)]. We explain that the statement of our convergence results can be simplified in this case and demonstrate by a toy example how to apply the results numerically in a Monte Carlo framework.

2. WEAK AND STRONG L^2 -CONVERGENCE VIA DISCRETE S -TRANSFORMS

In this section, we study strong and weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of a sequence (X^n) of random variables, where X^n is $\mathcal{F}^n := \sigma(\xi_i^n, i \in \mathbb{N})$ -measurable, to an \mathcal{F} -measurable X . As a main result of this section (Theorem 1), we provide an equivalent criterion for this convergence, which only requires to compute a family of $L^2(\Omega, \mathcal{F}^n, P)$ -inner products (hence, expectations which involve functionals of the discrete time noise $(\xi_i^n)_{i \in \mathbb{N}}$ only) and their limits as n tends to infinity. Before doing so, let us recall that B^n can be constructed via a Skorokhod embedding of the random walk

$$\left(\sum_{i=1}^j \xi_i \right)_{j \in \mathbb{N}}, \quad \xi_1, \xi_2, \dots \text{ independent and with the same distribution as } \xi,$$

into the rescaled Brownian motion $(\sqrt{n}B_{t/n})_{t \geq 0}$. In this way, one obtains, for every $n \in \mathbb{N}$, a sequence of stopping times $(\tau_i^n)_{i \in \mathbb{N}_0}$ with respect to the augmentation of the filtration generated by B such that

$$B^n := \left(B_{\tau_{[nt]}^n} \right)_{t \geq 0} \quad (2)$$

has the same distribution as $(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i)_{t \geq 0}$ and converges to B uniformly on compacts in probability (see e.g. [Mörters and Peres (2010), Lemma 5.24 (b)]).

We now introduce the S -transform simultaneously in the continuous time setting and the discrete time setting, which turns out to be the key tool for the proofs of our limit theorems. Recall, that the mapping $\mathbf{1}_{(0,t]} \mapsto B_t$ can be extended to a continuous linear mapping from $L^2([0, \infty))$

to $L^2(\Omega, \mathcal{F}, P)$, which is known as the *Wiener integral*. We denote the Wiener integral of a function $f \in L^2([0, \infty))$ by $I(f)$. The *discrete Wiener integral* is given by

$$I^n(f^n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} f^n(i) \xi_i^n.$$

Here, the discrete time function f^n is a member of

$$L_n^2(\mathbb{N}) := \left\{ f^n : \mathbb{N} \rightarrow \mathbb{R} : \|f^n\|_{L_n^2(\mathbb{N})}^2 := \frac{1}{n} \sum_{i=1}^{\infty} (f^n(i))^2 < \infty \right\},$$

which obviously ensures that the series $I^n(f^n)$ converges (strongly) in $L^2(\Omega, \mathcal{F}^n, P)$.

The *Wick exponential* is, by definition, the stochastic exponential of a Wiener integral $I(f)$, i.e.,

$$\exp^\diamond(I(f)) := \exp\left(I(f) - \frac{1}{2} \int_0^\infty f^2(s) ds\right).$$

Hence, its discrete counterpart, the *discrete Wick exponential*, is given by

$$\exp^{\diamond_n}(I^n(f^n)) := \prod_{i=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}} f^n(i) \xi_i^n\right). \quad (3)$$

In particular, by Fatou's lemma and the estimate $1 + x \leq \exp(x)$,

$$\mathbb{E}[(\exp^{\diamond_n}(I^n(f^n)))^2] \leq \exp(\|f^n\|_{L_n^2(\mathbb{N})}^2) < \infty. \quad (4)$$

Notice also that

$$\begin{aligned} \exp^{\diamond_n}(I^n(f^n)) &= 1 + \sum_{i=1}^{\infty} (\exp^{\diamond_n}(I^n(f^n \mathbf{1}_{[1,i]})) - \exp^{\diamond_n}(I^n(f^n \mathbf{1}_{[1,i-1]}))) \\ &= 1 + \sum_{i=1}^{\infty} f^n(i) \exp^{\diamond_n}(I^n(f^n \mathbf{1}_{[1,i-1]})) \frac{\xi_i^n}{\sqrt{n}}, \end{aligned} \quad (5)$$

which is the discrete counterpart of the Doléans-Dade equation.

We finally recall that, for every $X \in L^2(\Omega, \mathcal{F}, P)$ and $f \in L^2([0, \infty))$, the *S-transform* is defined as

$$(SX)(f) := \mathbb{E}[X \exp^\diamond(I(f))].$$

Analogously, for every $X^n \in L^2(\Omega, \mathcal{F}^n, P)$ and $f^n \in L_n^2(\mathbb{N})$, we introduce the *discrete S-transform* as

$$(S^n X^n)(f^n) := \mathbb{E}[X^n \exp^{\diamond_n}(I^n(f^n))].$$

We emphasize that the *S-transform* is a powerful tool in the white noise analysis, see, e.g., [Kuo (1996)], and has been successfully applied in the theory of stochastic partial differential equations, see [Holden et al. (2010)]. To the best of our knowledge the discrete *S-transform* has, however, not been studied in the literature.

Let us next denote by \mathcal{E} the set of step functions on left half-open intervals, i.e., functions of the form

$$g(x) = \sum_{j=1}^m a_j \mathbf{1}_{(b_j, c_j]}(x), \quad m \in \mathbb{N}, a_j, b_j, c_j \in \mathbb{R}.$$

As the set of Wick exponentials of step functions $\{\exp^\diamond(I(g)), g \in \mathcal{E}\}$ is total in $L^2(\Omega, \mathcal{F}, P)$, see e.g. [Janson (1997), Corollary 3.40], every $L^2(\Omega, \mathcal{F}, P)$ -random variable is uniquely determined by its *S-transform*. More precisely, if for $X, Y \in L^2(\Omega, \mathcal{F}, P)$, $(SX)(g) = (SY)(g)$ for every $g \in \mathcal{E}$, then $X = Y$ P -almost surely. We define the discretization of a step function $g \in \mathcal{E}$ as

$$\check{g}^n = (\check{g}^n(1), \check{g}^n(2), \dots) := (g(1/n), g(2/n), \dots),$$

and notice that

$$\{\check{g}^n : g \in \mathcal{E}\} \subset L_n^2(\mathbb{N})$$

is the dense subspace of discrete time functions with finite support.

The convergence results of integral and derivative operators in this paper rely on the following characterization of $L^2(\Omega, \mathcal{F}, P)$ -convergence in terms of convergence of the discrete S -transform to the continuous S -transform.

Theorem 1. *Suppose $X, X^n \in L^2(\Omega, \mathcal{F}, P)$ for every $n \in \mathbb{N}$, with X^n being \mathcal{F}^n -measurable. Then the following assertions are equivalent as n tends to infinity:*

- (i) $X^n \rightarrow X$ strongly (resp. weakly) in $L^2(\Omega, \mathcal{F}, P)$.
- (ii) $(S^n X^n)(\check{g}^n) \rightarrow (SX)(g)$ for every $g \in \mathcal{E}$, and additionally $\mathbb{E}[(X^n)^2] \rightarrow \mathbb{E}[X^2]$ in the case of strong convergence (resp. $\sup_{n \in \mathbb{N}} \mathbb{E}[(X^n)^2] < \infty$ in the case of weak convergence).

Moreover, in the case of strong convergence, (i) is also equivalent to

- (iii) $(X^n, \exp^{\diamond n}(I^n(\check{g}^n))) \rightarrow (X, \exp^{\diamond}(I(g)))$ in distribution for every $g \in \mathcal{E}$, and $((X^n)^2)_{n \in \mathbb{N}}$ is uniformly integrable.

Remark 2. *Note, that $X \in L^2(\Omega, \mathcal{F}, P)$ is, of course, not determined by its univariate distribution, but it is uniquely determined by all the bivariate distributions of $(X, e^{\diamond I(g)})$, $g \in \mathcal{E}$, in view of the injectivity of the S -transform. This observation motivates that the characterization of strong $L^2(\Omega, \mathcal{F}, P)$ -convergence via convergence in distribution in item (iii) of Theorem 1 can hold.*

In view of Lemma 4 below, the proof of Theorem 1 can be reduced to the following strong L^2 -convergence result for (discrete) Wick exponentials.

Proposition 3. *Suppose $g \in \mathcal{E}$. Then, we have strongly in $L^2(\Omega, \mathcal{F}, P)$, as n tends to infinity:*

$$\exp^{\diamond n}(I^n(\check{g}^n)) \rightarrow \exp^{\diamond}(I(g)).$$

These type of convergence results for stochastic exponentials are somewhat standard and can be obtained in a much more general context by applying weak convergence results for stochastic differential equations, see, e.g., [Avram (1988), Kurtz and Protter (1991)] and the references therein. For sake of completeness, we here provide an elementary proof.

Proof. Let

$$g = \sum_{j=1}^m a_j \mathbf{1}_{(b_j, c_j]} \in \mathcal{E}.$$

We denote by C, N constants in \mathbb{N} such that g is bounded by C and has support in $[0, N]$. Decomposing

$$\begin{aligned} & \mathbb{E} \left[(\exp^{\diamond}(I(g)) - \exp^{\diamond n}(I^n(\check{g}^n)))^2 \right] \\ &= \mathbb{E} \left[(\exp^{\diamond}(I(g)))^2 \right] - 2\mathbb{E} [\exp^{\diamond n}(I^n(\check{g}^n)) \exp^{\diamond}(I(g))] + \mathbb{E} \left[(\exp^{\diamond n}(I^n(\check{g}^n)))^2 \right], \end{aligned}$$

it suffices to show

- (i) $\lim_{n \rightarrow \infty} \mathbb{E} \left[(\exp^{\diamond n}(I^n(\check{g}^n)))^2 \right] = \mathbb{E} \left[(\exp^{\diamond}(I(g)))^2 \right],$
- (ii) $\exp^{\diamond n}(I^n(\check{g}^n)) \rightarrow \exp^{\diamond}(I(g))$ in probability,

because under (i) the integrand in the second term on the right-hand side is uniformly integrable.

(i) Due to $p < [q] \leq r \Leftrightarrow [p] < q \leq [r]$ for all $p, q, r \in \mathbb{R}$, we obtain for every $t \in (0, \infty)$,

$$\check{g}^n(\lceil nt \rceil) = \sum_{j=1}^m a_j \mathbf{1}_{([b_j n]/n, [c_j n]/n]}(t). \quad (6)$$

Hence,

$$\|g - \check{g}^n(\lceil n \cdot \rceil)\|_{L^2([0, \infty))} \leq \sqrt{2} \left(\sum_{j=1}^m |a_j| \right) \frac{1}{\sqrt{n}} \rightarrow 0, \quad (7)$$

and in particular,

$$\sum_{i=1}^{Nn} (\check{g}^n(i))^2 \frac{1}{n} = \|\check{g}^n(\lceil n \cdot \rceil)\|_{L^2([0, \infty))}^2 \rightarrow \int_0^\infty g(s)^2 ds.$$

Thus, by the independence of the centered random variables $(\xi_i^n)_{i \in \mathbb{N}}$ with unit variance and taking the boundedness of g into account, we get

$$\begin{aligned} \mathbb{E}[(\exp^{\diamond n}(I^n(\check{g}^n)))^2] &= \prod_{i=1}^{Nn} \mathbb{E}\left[\left(1 + \frac{1}{\sqrt{n}} \check{g}^n(i) \xi_i^n\right)^2\right] = \prod_{i=1}^{Nn} \left(1 + \frac{1}{n} (\check{g}^n(i))^2\right) \\ &\rightarrow \exp\left(\int_0^\infty g(s)^2 ds\right) = \mathbb{E}[(\exp^\diamond(I(g)))^2]. \end{aligned}$$

(ii) In order to treat the large jumps of B^n and the small ones separately, we consider

$$\xi_i^{n,1} := \xi_i^n \mathbf{1}_{\{|\xi_i^n| \leq \frac{\sqrt{n}}{2C}\}}, \quad \xi_i^{n,2} := \xi_i^n \mathbf{1}_{\{|\xi_i^n| > \frac{\sqrt{n}}{2C}\}},$$

cp. also [Sottinen (2001)]. Then,

$$\exp^{\diamond n}(I^n(\check{g}^n)) = \prod_{i=1}^{Nn} \left(1 + \frac{1}{\sqrt{n}} \check{g}^n(i) \xi_i^{n,1}\right) \prod_{i=1}^{Nn} \left(1 + \frac{1}{\sqrt{n}} \check{g}^n(i) \xi_i^{n,2}\right) =: E^{n,1} \cdot E^{n,2}$$

We note that, for every $\epsilon > 0$, by the independence of $(\xi_i^n)_{i \in \mathbb{N}}$,

$$P\left(\left\{\sup_{i=1, \dots, Nn} \frac{|\xi_i^n|}{\sqrt{n}} > \epsilon\right\}\right) = 1 - \left(1 - \frac{P(\{|\xi| > \epsilon\sqrt{n}\})Nn}{Nn}\right)^{Nn} \rightarrow 0, \quad (8)$$

because, by square-integrability of ξ , $P(\{|\xi| > \epsilon\sqrt{n}\})n \rightarrow 0$, see, e.g., [Shiryaev (1996), p. 208]. Hence, for every $\epsilon > 0$,

$$P(\{|E^{n,2} - 1| \geq \epsilon\}) \leq P(\{\sup_{i=1, \dots, Nn} |\xi_i^{n,2}| > 0\}) = P\left(\left\{\sup_{i=1, \dots, Nn} \frac{|\xi_i^n|}{\sqrt{n}} > 1/(2C)\right\}\right) \rightarrow 0,$$

i.e., $(E^{n,2})_{n \in \mathbb{N}}$ converges to 1 in probability.

By construction, each factor in $E^{n,1}$ is larger than $1/2$. Applying a Taylor expansion to the logarithm, thus, yields

$$\log E^{n,1} = \sum_{i=1}^{Nn} \check{g}^n(i) \frac{\xi_i^{n,1}}{\sqrt{n}} - \frac{1}{2} \sum_{i=1}^{Nn} (\check{g}^n(i))^2 \frac{(\xi_i^{n,1})^2}{n} + R_n$$

with a remainder term satisfying

$$|R_n| \leq \frac{8C}{3} \left(\sup_{j=1, \dots, Nn} \frac{|\xi_j^n|}{\sqrt{n}}\right) \sum_{i=1}^{Nn} (\check{g}^n(i))^2 \frac{(\xi_i^{n,1})^2}{n}.$$

It, thus, suffices to show

$$(iii) \quad \sum_{i=1}^{Nn} \check{g}^n(i) \frac{\xi_i^{n,1}}{\sqrt{n}} \rightarrow I(g) \text{ in probability,}$$

$$(iv) \quad \sum_{i=1}^{Nn} (\check{g}^n(i))^2 \frac{(\xi_i^{n,1})^2}{n} \rightarrow \int_0^\infty g(s)^2 ds \text{ in probability.}$$

Indeed, by (8), the remainder term then vanishes in probability as n tends to infinity, and, thus,

$$E^{n,1} \rightarrow \exp\left(I(g) - \frac{1}{2} \int_0^\infty g(s)^2 ds\right) \text{ in probability.}$$

The same argument, which was applied for the convergence of $E^{n,2}$, shows that we can (and shall) replace $\xi_i^{n,1}$ by ξ_i^n in (iii) and (iv). However, by (1) and (6),

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{Nn} \check{g}^n(i) \frac{\xi_i^n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \sum_{j=1}^m a_j (B_{c_j}^n - B_{b_j}^n) = \sum_{j=1}^m a_j (B_{c_j} - B_{b_j}) = I(g), \text{ in probability.}$$

Finally, by the law of large numbers, $\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} (\xi_i^n)^2$ converges to t in probability for every $t \geq 0$, and, hence, by (6),

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{Nn} (\check{g}^n(i))^2 \frac{(\xi_i^n)^2}{n} = \sum_{j=1}^m a_j^2 (c_j - b_j) = \int_0^\infty g(s)^2 ds, \text{ in probability.}$$

□

The following simple lemma from functional analysis turns out to be useful.

Lemma 4. *Suppose H is a Hilbert space, A is an arbitrary index set, $\{x^a, a \in A\}$ is total in H , and, for every $a \in A$, $(x_n^a)_{n \in \mathbb{N}}$ is a sequence in H which converges strongly in H to x^a . Then, the following are equivalent, as n tends to infinity:*

- (i) $x^n \rightarrow x$ strongly (resp. weakly) in H .
- (ii) $\langle x_n, x_n^a \rangle_H \rightarrow \langle x, x^a \rangle_H$ for every $a \in A$, and additionally $\|x_n\|_H \rightarrow \|x\|_H$ in the case of strong convergence (resp. $\sup_{n \in \mathbb{N}} \|x_n\|_H < \infty$ in the case of weak convergence).

Proof. Firstly, we observe that $\sup_{n \in \mathbb{N}} \|x_n\|_H$ is finite, either by weak convergence [Yosida (1995), Theorem V.1.1] in (i) or by assumption (ii). Thus, for every $a \in A$, by the strong convergence of (x_n^a) to x^a ,

$$|\langle x_n, x_n^a \rangle_H - \langle x_n, x^a \rangle_H| = |\langle x_n, x_n^a - x^a \rangle_H| \leq \sup_{m \in \mathbb{N}} \|x_m\|_H \|x_n^a - x^a\|_H \rightarrow 0. \quad (9)$$

Let us treat the case of weak convergence: If (i) holds, the term $\langle x_n, x^a \rangle_H$ in (9) converges to $\langle x, x^a \rangle_H$, and then so does $\langle x_n, x_n^a \rangle_H$, which implies (ii). Conversely, if (ii) holds, the first term $\langle x_n, x_n^a \rangle_H$ in (9) tends to $\langle x, x^a \rangle_H$, and then so does $\langle x_n, x^a \rangle_H$, which yields (i) in view of [Yosida (1995), Theorem V.1.3]. The case of strong convergence is an immediate consequence, as, in a Hilbert space, strong convergence is equivalent to weak convergence and convergence of the norms [Yosida (1995), Theorem V.1.8]. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1. ‘(i) \Leftrightarrow (ii)’: Proposition 3 and Lemma 4 apply immediately in view of the definition of the (discrete) S -transform, and as the set of Wick exponentials of step functions $\{\exp^\diamond(I(g)), g \in \mathcal{E}\}$ is total in $L^2(\Omega, \mathcal{F}, P)$.

‘(i) with strong convergence \Rightarrow (iii)’: This is a direct consequence of Proposition 3 and the assumed strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of (X_n) .

‘(iii) \Rightarrow (ii) with strong convergence’: By (iii) and the continuous mapping theorem, the sequence $(X^n \exp^{\diamond n}(I^n(\check{g}^n)))$ converges in distribution to $X \exp^\diamond(I(g))$. Moreover, this sequence is uniformly integrable, because so are the sequences $(|X^n|^2)$ by assumption and $(|\exp^{\diamond n}(I^n(\check{g}^n))|^2)$ by Proposition 3. Hence,

$$(S^n X^n)(\check{g}^n) = \mathbb{E}[X^n \exp^{\diamond n}(I^n(\check{g}^n))] \rightarrow \mathbb{E}[X \exp^\diamond I(g)] = (SX)(g).$$

Moreover, thanks to the uniform integrability of $((X^n)^2)$ and the convergence in distribution $X^n \xrightarrow{d} X$, we have $\mathbb{E}[(X^n)^2] \rightarrow \mathbb{E}[X^2]$. This completes the proof of (ii) with strong convergence. □

We close this section with an example.

Example 5. (i) In this example, we provide a simple proof, that, for every $X \in L^2(\Omega, \mathcal{F}, P)$, $X^n := \mathbb{E}[X|\mathcal{F}^n]$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$. Indeed, by Proposition 3, for every $g \in \mathcal{E}$,

$$(S^n X^n)(\check{g}^n) = \mathbb{E}[\mathbb{E}[X|\mathcal{F}^n] \exp^{\diamond n}(I^n(\check{g}^n))] = \mathbb{E}[X \exp^{\diamond n}(I^n(\check{g}^n))] \rightarrow \mathbb{E}[X \exp^{\diamond}(I(g))] = (SX)(g).$$

As $\mathbb{E}[(X^n)^2] \leq \mathbb{E}[X^2]$, Theorem 1 implies weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of (X^n) to X . The same theorem finally yields strong $L^2(\Omega, \mathcal{F}, P)$ -convergence, since, by the already established weak convergence,

$$\mathbb{E}[(X^n)^2] = \mathbb{E}[\mathbb{E}[X^n|\mathcal{F}^n] X] = \mathbb{E}[X^n X] \rightarrow \mathbb{E}[X^2].$$

We note that this result can alternatively be derived by the uniform integrability of $((X^n)^2)$ via the concept of convergence of filtrations making use of [Coquet et al. (2001), Proposition 2].

(ii) Denote by $(\mathcal{F}_t)_{t \geq 0}$ the augmented Brownian filtration and let $\mathcal{F}_i^n = \sigma(\xi_1^n, \dots, \xi_i^n)$. We assume $X \in L^2(\Omega, \mathcal{F}_T, P)$. Then, one can always approximate X by a sequence (X_T^n) strongly in $L^2(\Omega, \mathcal{F}, P)$, where X_T^n is measurable with respect to $\mathcal{F}_{[nT]}^n$. Indeed, take any sequence (X^n) of \mathcal{F}^n -measurable random variables which converges strongly in $L^2(\Omega, \mathcal{F}, P)$ to X , and define $X_T^n = \mathbb{E}[X^n|\mathcal{F}_{[nT]}^n]$. Then, for every $g \in \mathcal{E}$, by Proposition 3,

$$\begin{aligned} (S^n X_T^n)(\check{g}^n) &= \mathbb{E} \left[X^n \prod_{i=1}^{[nT]} \left(1 + \frac{1}{\sqrt{n}} g(i/n) \xi_i^n \right) \right] = \mathbb{E} \left[X^n \exp^{\diamond n}(I^n((g \mathbf{1}_{(0,T]}))^n) \right] \\ &\rightarrow \mathbb{E} [X \exp^{\diamond}(I(g \mathbf{1}_{(0,T]}))] = \mathbb{E} [X \mathbb{E}[\exp^{\diamond}(I(g))|\mathcal{F}_T]] = (SX)(g). \end{aligned}$$

Moreover,

$$\sup_{n \in \mathbb{N}} \mathbb{E}[(X_T^n)^2] \leq \sup_{n \in \mathbb{N}} \mathbb{E}[(X^n)^2] < \infty.$$

Hence, (X_T^n) converges weakly in $L^2(\Omega, \mathcal{F}, P)$ to X by Theorem 1. Then, strong $L^2(\Omega, \mathcal{F}, P)$ -convergence follows by Theorem 1 as well, because

$$\mathbb{E}[(X_T^n)^2] = \mathbb{E}[X_T^n X] + \mathbb{E}[X_T^n (X^n - X)] \rightarrow \mathbb{E}[X^2].$$

3. WEAK L^2 -APPROXIMATION OF THE SKOROKHOD INTEGRAL AND THE MALLIAVIN DERIVATIVE

In this section, we first discuss weak L^2 -approximations of the Skorokhod integral and the Malliavin derivative via appropriate discrete-time counterparts. We then show how to lift these results from weak convergence to strong convergence via duality under appropriate conditions which can be formulated in terms of the discrete-time approximations.

While most presentations of Malliavin calculus first introduce the Malliavin derivative and then define the Skorokhod integral as adjoint operator of the Malliavin derivative, we shall here employ the following equivalent characterization of the Skorokhod integral in terms of the S -transform, cp. [Janson (1997), Theorem 16.46, Theorem 16.50].

Definition 6. $Z \in L^2(\Omega \times [0, \infty)) := L^2(\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}([0, \infty)), P \otimes \lambda_{[0, \infty)})$ is said to belong to the domain $D(\delta)$ of the Skorokhod integral, if there is an $X \in L^2(\Omega, \mathcal{F}, P)$ such that for every $g \in \mathcal{E}$

$$(SX)(g) = \int_0^\infty (SZ_t)(g)g(t)dt.$$

In this case, X is uniquely determined and $\delta(Z) := X$ is called the Skorokhod integral of Z .

For the discrete-time approximation we first introduce the space

$$L_n^2(\Omega \times \mathbb{N}) := \left\{ Z^n : \mathbb{N} \rightarrow L^2(\Omega, \mathcal{F}^n, P), \ \|Z^n\|_{L_n^2(\Omega \times \mathbb{N})}^2 := \frac{1}{n} \sum_{i=1}^\infty \mathbb{E}[(Z_i^n)^2] < \infty \right\}.$$

Moreover, we recall the definitions

$$\mathcal{F}^n := \sigma(\xi_j^n, j \in \mathbb{N}), \quad \mathcal{F}_M^n := \sigma(\xi_1^n, \dots, \xi_M^n),$$

and introduce the shorthand notations

$$\mathcal{F}_{-i}^n := \sigma(\xi_j^n, j \in \mathbb{N} \setminus \{i\}), \quad \mathcal{F}_{M,-i}^n := \sigma(\xi_j^n, j \in \{1, \dots, M\} \setminus \{i\}).$$

Definition 7. We say, $Z^n \in L_n^2(\Omega \times \mathbb{N})$ belongs to the domain $D(\delta^n)$ of the discrete Skorokhod integral, if

$$\delta^n(Z^n) := \lim_{M \rightarrow \infty} \sum_{i=1}^M \mathbb{E}[Z_i^n | \mathcal{F}_{M,-i}^n] \frac{\xi_i^n}{\sqrt{n}}. \quad (10)$$

exists strongly in $L^2(\Omega, \mathcal{F}, P)$. If this is the case, $\delta^n(Z^n)$ is called the discrete Skorokhod integral of Z^n .

We note that, by the independence of $\mathbb{E}[Z_i^n | \mathcal{F}_{M,-i}^n]$ and ξ_i^n , each summand on the right-hand side of (10) is indeed a member of $L^2(\Omega, \mathcal{F}, P)$. Moreover, the martingale convergence theorem implies that, for every $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $N \in \mathbb{N}$, $Z^n \mathbf{1}_{[1,N]} \in D(\delta^n)$ and

$$\delta^n(Z^n \mathbf{1}_{[1,N]}) = \sum_{i=1}^N \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \frac{\xi_i^n}{\sqrt{n}}. \quad (11)$$

Hence, the discrete Skorokhod integral is densely defined from $L_n^2(\Omega \times \mathbb{N})$ to $L^2(\Omega, \mathcal{F}, P)$. We will show in Proposition 14 below that it is a closed operator.

Remark 8. This definition of the discrete Skorokhod integral closely resembles the following Riemann-sum approximation of the Skorokhod integral by [Nualart and Pardoux (1988)], who show that under appropriate conditions on Z ,

$$\delta(Z \mathbf{1}_{[0,1]}) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \mathbb{E} \left[n \int_{i/n}^{(i+1)/n} Z_s ds \middle| (B_s, B_1 - B_r)_{0 \leq s \leq i/n \leq (i+1)/n \leq r \leq 1} \right] (B_{(i+1)/n} - B_{i/n})$$

strongly in $L^2(\Omega, \mathcal{F}, P)$.

As a first main result of this section we are going to show the following weak approximation theorem for Skorokhod integrals.

Theorem 9. Suppose $Z^n \in D(\delta^n)$ for every $n \in \mathbb{N}$, and $(Z_{[n,\cdot]}^n)_{n \in \mathbb{N}}$ converges to Z weakly in $L^2(\Omega \times [0, \infty))$. Then, the following assertions are equivalent:

- (i) $\sup_{n \in \mathbb{N}} \mathbb{E}[\delta^n(Z^n)^2] < \infty$.
- (ii) $Z \in D(\delta)$ and $(\delta^n(Z^n))$ converges to $\delta(Z)$ weakly in $L^2(\Omega, \mathcal{F}, P)$ as n tends to infinity.

As a first tool for the proof we state the discrete S -transform of a discrete Skorokhod integral.

Proposition 10. Suppose $Z^n \in D(\delta^n)$. Then, for every $g \in \mathcal{E}$,

$$(S^n \delta^n(Z^n))(\check{g}^n) = \frac{1}{n} \sum_{i=1}^{\infty} (S^n Z_i^n)(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}}) \check{g}^n(i).$$

This result is a special case of the more general Proposition 14 below, to which we refer the reader for the proof.

The second tool for the proof of Theorem 9 is the following variant of Theorem 1 for stochastic processes.

Theorem 11. Suppose $Z \in L^2(\Omega \times [0, \infty))$, $(Z^n)_{n \in \mathbb{N}}$ satisfies $Z^n \in L_n^2(\Omega \times \mathbb{N})$ for every $n \in \mathbb{N}$. Then the following assertions are equivalent as n tends to infinity:

- (i) $(Z_{[n,\cdot]}^n)$ converges strongly (resp. weakly) to Z in $L^2(\Omega \times [0, \infty))$.
- (ii) For every $g, h \in \mathcal{E}$

$$\frac{1}{n} \sum_{i=1}^{\infty} (S^n Z_i^n)(\check{g}^n) \check{h}^n(i) \rightarrow \int_0^{\infty} (SZ_s)(g)h(s)ds.$$

and, additionally, $\mathbb{E}[\int_0^\infty (Z_{[ns]}^n)^2 ds] \rightarrow \mathbb{E}[\int_0^\infty Z_s^2 ds]$ in the case of strong convergence (resp. $\sup_{n \in \mathbb{N}} \mathbb{E}[\int_0^\infty (Z_{[ns]}^n)^2 ds] < \infty$ in the case of weak convergence).

Moreover, in (ii), $\frac{1}{n} \sum_{i=1}^\infty (S^n Z_i^n)(\check{g}^n) \check{h}^n(i)$ can be replaced by $\frac{1}{n} \sum_{i=1}^\infty (S^n Z_i^n)(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}}) \check{h}^n(i)$.

Proof. We wish to apply Lemma 4 in order to prove the equivalence of (i) and (ii). As $L^2(\Omega \times [0, \infty)) = L^2(\Omega, \mathcal{F}, P) \otimes L^2([0, \infty))$ (with the tensor product in the sense of Hilbert spaces), the set $\{\exp^\diamond(I(g))h; g, h \in \mathcal{E}\}$ is total in $L^2(\Omega \times [0, \infty))$. In view of Proposition 3 and (7), $(\exp^{\diamond n}(I^n(\check{g}^n))\check{h}^n(\lceil n \cdot \rceil))_{n \in \mathbb{N}}$ converges to $\exp^\diamond(I(g))h$ strongly in $L^2(\Omega \times [0, \infty))$ for every $g, h \in \mathcal{E}$. As

$$\frac{1}{n} \sum_{i=1}^\infty (S^n Z_i^n)(\check{g}^n) \check{h}^n(i) = \left\langle Z_{\lceil n \cdot \rceil}^n, e^{\diamond n}(I^n(\check{g}^n)) \check{h}^n(\lceil n \cdot \rceil) \right\rangle_{L^2(\Omega \times [0, \infty))},$$

Lemma 4 applies indeed.

We finally note, that the ‘Moreover’-part of the assertion is an immediate consequence of the Cauchy-Schwarz inequality and the estimate

$$\begin{aligned} & \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n)) - \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}})) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}})) \right)^2 \right] \mathbb{E} \left[(\check{g}^n(i) \xi_i^n / \sqrt{n})^2 \right] \\ &\leq \exp(\|\check{g}^n\|_{L_n^2(\mathbb{N})}^2) \sup_{j \in \mathbb{N}} |g(j)|^2 / n \rightarrow 0, \end{aligned}$$

making use of (4) in the last line. \square

We are now ready to give the proof of Theorem 9.

Proof of Theorem 9. As the implication ‘(ii) \Rightarrow (i)’ is trivial, we only have to show the converse implication. To this end, note first that, by Proposition 10 and Theorem 11, for every $g \in \mathcal{E}$,

$$\lim_{n \rightarrow \infty} (S^n \delta^n(Z^n))(\check{g}^n) = \int_0^\infty (SZ_t)(g)g(t)dt. \quad (12)$$

As the sequence $(\delta^n(Z^n))_{n \in \mathbb{N}}$ is norm bounded by (i), it has a weakly convergent subsequence [Yosida (1995), Theorem V.2.1]. We denote its limit by X . Then, applying Theorem 1 and (12) along the subsequence, we obtain, for every $g \in \mathcal{E}$,

$$(SX)(g) = \int_0^\infty (SZ_t)(g)g(t)dt. \quad (13)$$

Hence, by Definition 6, $Z \in D(\delta)$ and $\delta(Z) = X$. Finally, by Theorem 1 and (12)–(13), weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(\delta^n(Z^n))_{n \in \mathbb{N}}$ to $\delta(Z)$ holds along the whole sequence, and not only along the subsequence. \square

We now turn to the weak approximation of the Malliavin derivative. Again, we apply a definition in terms of the S -transform, which we show to be equivalent to the more classical one in terms of the chaos decomposition in the Appendix.

Definition 12. A random variable $X \in L^2(\Omega, \mathcal{F}, P)$ is said to belong to the domain $\mathbb{D}^{1,2}$ of the Malliavin derivative, if there is a stochastic process $Z \in L^2(\Omega \times [0, \infty))$ such that for every $g, h \in \mathcal{E}$,

$$\int_0^\infty (SZ_s)(g)h(s)ds = \mathbb{E} \left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right].$$

In this case, Z is unique and $DX := Z$ is called the Malliavin derivative X .

For every $X \in L^2(\Omega, \mathcal{F}, P)$ we define the discretized Malliavin derivative of X at $j \in \mathbb{N}$ with respect to $(\xi_i^n)_{i \in \mathbb{N}}$ by

$$D_j^n X := \sqrt{n} \mathbb{E}[\xi_j^n X | \mathcal{F}_{-j}^n].$$

We observe that, for fixed j , D_j^n is a continuous linear operator from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{F}, P)$, because by Hölder's inequality for conditional expectations and the independence of the family $(\xi_i^n)_{i \in \mathbb{N}}$,

$$|D_j^n X|^2 \leq n \mathbb{E}[X^2 | \mathcal{F}_{-j}^n] \mathbb{E}[(\xi_j^n)^2 | \mathcal{F}_{-j}^n] = n \mathbb{E}[X^2 | \mathcal{F}_{-j}^n].$$

We say that X belongs to the domain $\mathbb{D}_n^{1,2}$ of the discretized Malliavin derivative, if the process $D^n X := (D_i^n X)_{i \in \mathbb{N}}$ is a member of $L_n^2(\Omega \times \mathbb{N})$. In this case $D^n X$ is called the *discretized Malliavin derivative* of X with respect to $(\xi_i^n)_{i \in \mathbb{N}}$. As D_j^n is continuous for fixed j , it is easy to check that the discretized Malliavin derivative is a densely defined closed operator from $L^2(\Omega, \mathcal{F}, P)$ to $L_n^2(\Omega \times \mathbb{N})$.

In the following theorem and in the remainder of the paper we use the convention $Z_0^n = 0$ for $Z^n \in L_n^2(\Omega \times \mathbb{N})$.

Theorem 13. *Suppose $(X^n)_{n \in \mathbb{N}}$ converges to X weakly in $L^2(\Omega, \mathcal{F}, P)$ and $X^n \in \mathbb{D}_n^{1,2}$ for every $n \in \mathbb{N}$. Then, the following are equivalent:*

- (i) $\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[(D_i^n X^n)^2] < \infty$.
- (ii) $X \in \mathbb{D}^{1,2}$ and $(D_{[n, \cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to DX weakly in $L^2(\Omega \times [0, \infty))$.

The proof is prepared by two propositions. The first one contains the duality relation between the discrete Skorokhod integral and discretized Malliavin derivative.

Proposition 14. *For every $n \in \mathbb{N}$, the discrete Skorokhod integral is the adjoint operator of the discretized Malliavin derivative. In particular, δ^n is closed and, for every $X \in \mathbb{D}_n^{1,2}$ and $Z^n \in D(\delta^n)$,*

$$\frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n D_i^n X] = \mathbb{E}[\delta^n(Z^n)X].$$

We emphasize that, choosing $X = \exp^{\diamond n}(I^n(\check{g}^n))$, $g \in \mathcal{E}$, in Proposition 14, we obtain the assertion of Proposition 10. Indeed, we only have to note that, for every $f^n \in L_n^2(\mathbb{N})$,

$$D_i^n \exp^{\diamond n}(I^n(f^n)) = f^n(i) \exp^{\diamond n}(I^n(f^n \mathbf{1}_{\mathbb{N} \setminus \{i\}})).$$

Proof. Suppose first, that $Z^n \in D(\delta^n)$ and $X \in \mathbb{D}_n^{1,2}$. Then, for every $M \in \mathbb{N}$, and $i \in \mathbb{N}$,

$$\mathbb{E} \left[\left| \sqrt{n} \mathbb{E}[\xi_i^n X | \mathcal{F}_{M, -i}^n] \right|^2 \right] \leq \mathbb{E} \left[|D_i^n X|^2 \right].$$

Hence, by the martingale convergence theorem and dominated convergence,

$$\lim_{M \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[\left| \sqrt{n} \mathbb{E}[\xi_i^n X | \mathcal{F}_{M, -i}^n] - D_i^n X \right|^2 \right] = 0.$$

Consequently,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n D_i^n X] &= \lim_{M \rightarrow \infty} \frac{1}{n} \sum_{i=1}^M \mathbb{E}[Z_i^n \sqrt{n} \mathbb{E}[\xi_i^n X | \mathcal{F}_{M, -i}^n]] \\ &= \lim_{M \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^M \mathbb{E}[X \xi_i^n \mathbb{E}[Z_i^n | \mathcal{F}_{M, -i}^n]] \\ &= \lim_{M \rightarrow \infty} \mathbb{E} \left[X \sum_{i=1}^M \mathbb{E}[Z_i^n | \mathcal{F}_{M, -i}^n] \frac{\xi_i^n}{\sqrt{n}} \right] = \mathbb{E}[X \delta^n(Z^n)]. \end{aligned}$$

Conversely, suppose that Z^n is in the domain of the adjoint operator of the discretized Malliavin derivative, i.e., there is an $Y^n \in L^2(\Omega, \mathcal{F}, P)$ such that for every $X \in \mathbb{D}_n^{1,2}$,

$$\frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n D_i^n X] = \mathbb{E}[Y^n X]. \quad (14)$$

We first note that, by construction, $X \in \mathbb{D}_n^{1,2}$ if and only if $\mathbb{E}[X|\mathcal{F}^n] \in \mathbb{D}_n^{1,2}$, and, if this is the case, both random variables have the same discretized Malliavin derivative. Hence, applying the duality relation (14), with X and $\mathbb{E}[X|\mathcal{F}^n]$, we observe that, $Y^n = \mathbb{E}[Y^n|\mathcal{F}^n]$. Now suppose that $X \in L^2(\Omega, \mathcal{F}_M^n, P)$. Then $X \in \mathbb{D}_n^{1,2}$, $D_i^n X = \sqrt{n}\mathbb{E}[\xi_i^n X|\mathcal{F}_{M,-i}^n]$ for every $i \leq M$, and $D_i^n X = 0$ for $i > M$. Hence, (14) and the same manipulations as above imply

$$\mathbb{E}[Y^n X] = \mathbb{E} \left[X \sum_{i=1}^M \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n] \frac{\xi_i^n}{\sqrt{n}} \right],$$

i.e.

$$\mathbb{E}[Y^n|\mathcal{F}_M^n] = \sum_{i=1}^M \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n] \frac{\xi_i^n}{\sqrt{n}}.$$

By the martingale convergence theorem, $(\mathbb{E}[Y^n|\mathcal{F}_M^n])_{M \in \mathbb{N}}$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$ to $\mathbb{E}[Y^n|\mathcal{F}^n] = Y^n$. Hence, $Z^n \in D(\delta^n)$ and $\delta^n(Z^n) = Y^n$. Finally, closedness is a general property of adjoint operators, see, e.g., [Yosida (1995), p. 196]. \square

The next proposition is a consequence of the weak convergence result for discrete Skorokhod integrals in Theorem 9.

Proposition 15. *For every $g, h \in \mathcal{E}$,*

$$\lim_{n \rightarrow \infty} \delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n) = \exp^{\diamond}(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right)$$

strongly in $L^2(\Omega, \mathcal{F}, P)$.

Proof. Notice first that, for fixed $n \in \mathbb{N}$, $\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n \in D(\delta^n)$, because $\check{h}^n(i)$ vanishes, if i is sufficiently large. A direct computation, making use of (11), shows

$$\delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n) = \sum_{i=1}^\infty \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}})) \check{h}^n(i) \frac{\xi_i^n}{\sqrt{n}}.$$

For $i \neq j$ we obtain, by independence of $(\xi_k^n)_{k \in \mathbb{N}}$,

$$\mathbb{E} \left[\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{i\}})) \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{\mathbb{N} \setminus \{j\}})) \xi_i^n \xi_j^n \right] = \check{g}^n(i) \frac{1}{\sqrt{n}} \check{g}^n(j) \frac{1}{\sqrt{n}} \prod_{k \in \mathbb{N} \setminus \{i,j\}} \left(1 + \check{g}^n(k)^2 \frac{1}{n} \right).$$

Combining this with an analogous calculation for the case $i = j$ yields

$$\begin{aligned} \mathbb{E} \left[\left| \delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n) \right|^2 \right] &= \frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \check{h}^n(i) \check{g}^n(i) \check{h}^n(j) \check{g}^n(j) \prod_{k \in \mathbb{N} \setminus \{i,j\}} \left(1 + \check{g}^n(k)^2 \frac{1}{n} \right) \\ &\quad + \frac{1}{n} \sum_{i=1}^\infty \check{h}^n(i)^2 \prod_{k \in \mathbb{N} \setminus \{i\}} \left(1 + \check{g}^n(k)^2 \frac{1}{n} \right). \end{aligned}$$

As g and h are bounded with compact support, it is straightforward to check in view of (7) that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n) \right|^2 \right] = e^{\int_0^\infty g(s)^2 ds} \left(\left(\int_0^\infty h(s)g(s)ds \right)^2 + \int_0^\infty h(s)^2 ds \right). \quad (15)$$

Thus, $(\delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n))_{n \in \mathbb{N}}$ converges to $\delta(\exp^{\diamond}(I(g)) h)$ weakly in $L^2(\Omega, \mathcal{F}, P)$ by Theorem 9. The identity

$$\delta(\exp^{\diamond}(I(g)) h) = \exp^{\diamond}(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right)$$

can either be derived by a direct computation making use of the S -transform definition of the Skorokhod integral (Definition 6) or alternatively is a simple consequence of [Nualart (2006)],

Proposition 1.3.3] in conjunction with Definition 1.2.1 in the same reference. Applying the Cameron-Martin shift [Janson (1997), Theorem 14.1] twice, we observe

$$\begin{aligned} & \mathbb{E} \left[\left(\exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right)^2 \right] = e^{\int_0^\infty g(s)^2 ds} \mathbb{E} [\exp^\diamond(I(g)) I(h)^2] \\ &= e^{\int_0^\infty g(s)^2 ds} \mathbb{E} \left[\left(I(h) + \int_0^\infty g(s)h(s)ds \right)^2 \right] \\ &= e^{\int_0^\infty g(s)^2 ds} \left(\left(\int_0^\infty h(s)g(s)ds \right)^2 + \int_0^\infty h(s)^2 ds \right). \end{aligned}$$

Thanks to (15), this turns weak into strong convergence. \square

The proof of Theorem 13 is now analogous to that of Theorem 9.

Proof of Theorem 13. ‘(ii) \Rightarrow (i)’ is obvious, since $\frac{1}{n} \sum_{i=1}^\infty \mathbb{E}[(D_i^n X^n)^2] = \int_0^\infty \mathbb{E}[(D_{[n\cdot]}^n X^n)^2] ds$. ‘(i) \Rightarrow (ii)’: Notice first that, for every $g, h \in \mathcal{E}$, by Proposition 14 with $Z^n = \exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n$ and Proposition 15,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^\infty (S^n D_i^n X^n)(\check{g}^n) \check{h}^n(i) &= \lim_{n \rightarrow \infty} \mathbb{E}[X^n \delta^n(\exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n)] \\ &= \mathbb{E} \left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right], \end{aligned} \quad (16)$$

since (X^n) converges to X weakly in $L^2(\Omega, \mathcal{F}, P)$. As the sequence $(D_{[n\cdot]}^n X^n)_{n \in \mathbb{N}}$ is norm bounded in $L^2(\Omega \times [0, \infty))$ by (i), it has a weakly convergent subsequence. We denote its limit by Z . Applying (16) and Theorem 11 along this subsequence, we conclude

$$\int_0^\infty (SZ_s)(g)h(s)ds = \mathbb{E} \left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right]. \quad (17)$$

Hence, $X \in \mathbb{D}^{1,2}$ and $DX = Z$ by Definition 12. Finally, applying (16)–(17) and Theorem 11 along the whole sequence $(D_{[n\cdot]}^n X^n)_{n \in \mathbb{N}}$, shows that this sequence converges weakly in $L^2(\Omega \times [0, \infty))$ to DX . \square

In order to check the assumptions of Theorem 9, we consider the space $\mathbb{L}_n^{1,2}$, which consists of processes $Z^n \in L_n^2(\Omega \times \mathbb{N})$ such that $Z_i^n \in \mathbb{D}_n^{1,2}$ for every $i \in \mathbb{N}$ and

$$\frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E} [|D_j^n Z_i^n|^2] < \infty. \quad (18)$$

Proposition 16. *For every $n \in \mathbb{N}$, $\mathbb{L}_n^{1,2} \subset D(\delta^n)$ and, for $Z^n \in \mathbb{L}_n^{1,2}$,*

$$\delta^n(Z^n) = \sum_{i=1}^\infty \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \frac{\xi_i^n}{\sqrt{n}}, \quad (\text{strong } L^2(\Omega, \mathcal{F}, P)\text{-convergence}), \quad (19)$$

$$\mathbb{E} [(\delta^n(Z^n))^2] = \frac{1}{n} \sum_{i=1}^\infty \mathbb{E} [\mathbb{E} [Z_i^n | \mathcal{F}_{-i}^n]^2] + \frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E} [(D_i^n Z_j^n)(D_j^n Z_i^n)]. \quad (20)$$

In particular, in the context of Theorem 9, assertion (i) is equivalent to

$$(i') \sup_{n \in \mathbb{N}} \frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E} [(D_i^n Z_j^n)(D_j^n Z_i^n)] < \infty,$$

if we additionally assume that $Z^n \in \mathbb{L}_n^{1,2}$ for every $n \in \mathbb{N}$.

Proof. Fix $N_1 < N_2 \in \mathbb{N}$. Then,

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{i=N_1}^{N_2} \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \frac{\xi_i^n}{\sqrt{n}} \right)^2 \right] = \frac{1}{n} \sum_{i=N_1}^{N_2} \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n]^2 (\xi_i^n)^2 \right] \\ & + \frac{1}{n} \sum_{i,j=N_1, i \neq j}^{N_2} \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \mathbb{E}[Z_j^n | \mathcal{F}_{-j}^n] \xi_i^n \xi_j^n \right] = (I)_{N_1, N_2} + (II)_{N_1, N_2}. \end{aligned}$$

By the independence of the discrete-time noise $(\xi_i^n)_{i \in \mathbb{N}}$ and as the conditional expectation has norm 1,

$$(I)_{1, N} = \frac{1}{n} \sum_{i=1}^N \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n]^2 \right] \rightarrow \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n]^2 \right] < \infty, \quad N \rightarrow \infty, \quad (21)$$

and $(I)_{N_1, N_2} \rightarrow 0$ as N_1, N_2 tend to infinity. In order to treat $(II)_{N_1, N_2}$, we first note that for any random variable $X^n \in L^1(\Omega, \mathcal{F}^n, P)$ and $i \neq j \in \mathbb{N}$, by Fubini's theorem,

$$\mathbb{E} \left[\mathbb{E} \left[X^n | \mathcal{F}_{-i}^n \right] | \mathcal{F}_{-j}^n \right] = \mathbb{E} \left[\mathbb{E} \left[X^n | \mathcal{F}_{-j}^n \right] | \mathcal{F}_{-i}^n \right]. \quad (22)$$

Hence, for $i \neq j \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \mathbb{E}[Z_j^n | \mathcal{F}_{-j}^n] \xi_i^n \xi_j^n \right] = \mathbb{E} \left[\mathbb{E}[Z_i^n \xi_i^n | \mathcal{F}_{-i}^n] \mathbb{E}[Z_j^n \xi_j^n | \mathcal{F}_{-j}^n] \right] \\ & = \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[Z_i^n \xi_i^n | \mathcal{F}_{-i}^n] | \mathcal{F}_{-j}^n \right] Z_j^n \xi_j^n \right] = \mathbb{E} \left[\mathbb{E} \left[\mathbb{E}[Z_i^n \xi_i^n | \mathcal{F}_{-j}^n] | \mathcal{F}_{-i}^n \right] Z_j^n \xi_j^n \right] \\ & = \mathbb{E} \left[\mathbb{E}[Z_i^n \xi_i^n | \mathcal{F}_{-j}^n] \mathbb{E}[Z_j^n \xi_j^n | \mathcal{F}_{-i}^n] \right] = \frac{1}{n} \mathbb{E} \left[(D_i^n Z_j^n)(D_j^n Z_i^n) \right]. \end{aligned}$$

Consequently, by Young's inequality,

$$n \left| \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \mathbb{E}[Z_j^n | \mathcal{F}_{-j}^n] \xi_i^n \xi_j^n \right] \right| \leq \frac{1}{2} \mathbb{E} \left[(D_i^n Z_j^n)^2 \right] + \frac{1}{2} \mathbb{E} \left[(D_j^n Z_i^n)^2 \right].$$

The $\mathbb{L}_n^{1,2}$ -assumption, thus, ensures that

$$\lim_{N \rightarrow \infty} (II)_{1, N} = \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{\infty} \mathbb{E} \left[(D_i^n Z_j^n)(D_j^n Z_i^n) \right] < \infty$$

and $(II)_{N_1, N_2} \rightarrow 0$ as N_1, N_2 tend to infinity. Hence, by (11), the sequence $(\delta^n(Z^n \mathbf{1}_{[1, N]}))_{N \in \mathbb{N}}$ is Cauchy in $L^2(\Omega, \mathcal{F}, P)$. By the closedness of the discrete Skorokhod integral, $Z^n \in D(\delta^n)$ and we obtain $\mathbb{L}_n^{1,2} \subset D(\delta^n)$, (19) and (20). We finally suppose that the assumptions of Theorem 9 are in force and that $Z^n \in \mathbb{L}_n^{1,2}$ for every $n \in \mathbb{N}$. Then,

$$\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n]^2 \right] < \infty,$$

because of the assumed weak convergence of the sequence $(Z^n)_{n \in \mathbb{N}}$. Thus, the sequence $(\delta^n(Z^n))_{n \in \mathbb{N}}$ is norm bounded in $L^2(\Omega, \mathcal{F}, P)$, if and only if (i') holds. \square

As a consequence of the previous proposition, we obtain the following strong $L^2(\Omega, \mathcal{F}, P)$ -convergence results to the Malliavin derivative.

Theorem 17. Suppose $(X^n)_{n \in \mathbb{N}}$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$. Moreover assume that $X^n \in \mathbb{D}_n^{2,2}$ for every $n \in \mathbb{N}$, i.e.

$$\frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[(D_i^n X)^2 \right] + \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{\infty} \mathbb{E} \left[(D_j^n D_i^n X)^2 \right] < \infty.$$

Then, the following assertions are equivalent:

$$(i) \sup_{n \in \mathbb{N}} \left(\frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[(D_i^n X)^2 \right] + \frac{1}{n^2} \sum_{i,j=1, i \neq j}^{\infty} \mathbb{E} \left[(D_j^n D_i^n X)^2 \right] \right) < \infty.$$

- (ii) $X \in \mathbb{D}^{1,2}$, $DX \in D(\delta)$, $(D_{[n,\cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to DX strongly in $L^2(\Omega \times [0, \infty))$, and $(\delta^n(D^n X^n))_{n \in \mathbb{N}}$ converges to $\delta(DX)$ weakly in $L^2(\Omega, \mathcal{F}, P)$.

Remark 18. Recall that $L = -\delta \circ D$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup, see [Nualart (2006), Section 1.4], and is sometimes called Ornstein-Uhlenbeck operator (cf. also [Janson (1997), Example 4.7]). So the previous theorem provides, at the same time, sufficient conditions for the strong convergence to the Malliavin derivative and the weak convergence to the Ornstein-Uhlenbeck operator.

Proof. Let $Z_i^n = D_i^n X^n$. Then, $X^n \in \mathbb{D}_n^{2,2}$ implies $Z^n \in \mathbb{L}_n^{1,2}$. Note that, for $i \neq j$, by (22),

$$D_j^n Z_i^n = D_j^n D_i^n X = D_i^n D_j^n X = D_i^n Z_j^n,$$

i.e. $(D_j^n Z_i^n)(D_i^n Z_j^n) = (D_j^n D_i^n X)^2$. Hence, by Theorem 13 and Theorem 9 in conjunction with Proposition 16, assertion (i) is equivalent to

- (ii') $X \in \mathbb{D}^{1,2}$, $DX \in D(\delta)$, $(D_{[n,\cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to DX weakly in $L^2(\Omega \times [0, \infty))$, and $(\delta^n(D^n X^n))_{n \in \mathbb{N}}$ converges to $\delta(DX)$ weakly in $L^2(\Omega, \mathcal{F}, P)$.

So we only need to show that under (ii') the convergence of $(D_{[n,\cdot]}^n X^n)_{n \in \mathbb{N}}$ to DX holds true in the strong topology. However, by the duality relation in Proposition 14, the weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(\delta^n(D^n X^n))_{n \in \mathbb{N}}$ and the strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(X^n)_{n \in \mathbb{N}}$,

$$\int_0^\infty \mathbb{E}[(D_{[n,\cdot]}^n X^n)^2] dt = \mathbb{E}[\delta^n(D^n X^n)X^n] \rightarrow \mathbb{E}[\delta(DX)X] = \int_0^\infty \mathbb{E}[(D_t X)^2] dt,$$

making use of the continuous time duality between Skorokhod integral and Malliavin derivative in the last step. \square

The analogous result for the Skorokhod integral reads as follows.

Theorem 19. Suppose $(Z_{[n,\cdot]}^n)_{n \in \mathbb{N}}$ converges strongly to Z in $L^2(\Omega \times [0, \infty))$ and assume that $Z^n \in \mathbb{L}_n^{2,2}$, i.e., for every $n \in \mathbb{N}$,

$$\frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E}[|D_i^n Z_j^n|^2] + \frac{1}{n^3} \sum_{i,j,k=1, |\{i,j,k\}|=3}^\infty \mathbb{E}[|D_i^n D_j^n Z_k^n|^2] < \infty.$$

Then the following assertions are equivalent:

- (i) $\sup_{n \in \mathbb{N}} \left(\frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E}[(D_i^n Z_j^n)(D_j^n Z_i^n)] \right) < \infty$ and
- $$\sup_{n \in \mathbb{N}} \left(\frac{1}{n^2} \sum_{i,j=1, i \neq j}^\infty \mathbb{E}[\mathbb{E}[D_i^n Z_j^n | \mathcal{F}_{-j}^n]^2] + \frac{1}{n^3} \sum_{i,j,k=1, |\{i,j,k\}|=3}^\infty \mathbb{E}[(D_i^n D_j^n Z_k^n)(D_k^n D_j^n Z_i^n)] \right) < \infty.$$
- (ii) $Z \in D(\delta)$, $\delta(Z) \in \mathbb{D}^{1,2}$, $(\delta^n(Z^n))_{n \in \mathbb{N}}$ converges to $\delta(Z)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ and $(D_{[n,\cdot]}^n \delta^n(Z^n))_{n \in \mathbb{N}}$ converges to $D\delta(Z)$ weakly in $L^2(\Omega \times [0, \infty))$.

As a preparation we explain how to compute the discretized Malliavin derivative of a discrete Skorokhod integral, which is analogous to the continuous-time situation, cp. e.g. [Nualart (2006), Proposition 1.3.8].

Proposition 20. Suppose $Z^n \in \mathbb{L}_n^{1,2}$. Then $(D_i^n Z^n) \mathbf{1}_{\mathbb{N} \setminus \{i\}} \in D(\delta^n)$ for every $i \in \mathbb{N}$, and

$$D_i^n \delta^n(Z^n) = \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] + \delta^n(D_i^n Z^n \mathbf{1}_{\mathbb{N} \setminus \{i\}}).$$

Proof. By (19) and the continuity of D_i^n ,

$$D_i^n \delta^n(Z^n) = D_i^n \left(\mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] \frac{\xi_i^n}{\sqrt{n}} \right) + \sum_{j=1, j \neq i}^\infty D_i^n \left(\mathbb{E}[Z_j^n | \mathcal{F}_{-j}^n] \frac{\xi_j^n}{\sqrt{n}} \right),$$

(including the strong convergence of the series on the right-hand side in $L^2(\Omega, \mathcal{F}, P)$). By (22), for $i \neq j$,

$$\mathbb{E}[\xi_i^n \mathbb{E}[Z_j^n | \mathcal{F}_{-j}^n] \xi_j^n | \mathcal{F}_{-i}^n] = \mathbb{E}[\mathbb{E}[\xi_i^n Z_j^n | \mathcal{F}_{-j}^n] | \mathcal{F}_{-i}^n] \xi_j^n = \mathbb{E}[\mathbb{E}[\xi_i^n Z_j^n | \mathcal{F}_{-i}^n] | \mathcal{F}_{-j}^n] \xi_j^n.$$

Moreover,

$$\mathbb{E}[(\xi_i^n)^2 \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] | \mathcal{F}_{-i}^n] = \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n].$$

Hence,

$$D_i^n \delta^n(Z^n) = \mathbb{E}[Z_i^n | \mathcal{F}_{-i}^n] + \sum_{j=1, j \neq i}^{\infty} \mathbb{E}[D_i^n Z_j^n | \mathcal{F}_{-j}^n] \frac{\xi_j^n}{\sqrt{n}},$$

and the closedness of the discrete Skorokhod integral concludes. \square

Proof of Theorem 19. The $\mathbb{L}_n^{2,2}$ -assumption guarantees that, for every $i \in \mathbb{N}$, $(D_i^n Z^n) \mathbf{1}_{\mathbb{N} \setminus \{i\}} \in \mathbb{L}_n^{1,2}$. As $(Z_{[n \cdot]}^n)_{n \in \mathbb{N}}$ is norm bounded in $L^2(\Omega \times [0, \infty))$ by the assumed strong convergence to Z , we observe in view of Propositions 16 and 20 that (i) is equivalent to

$$(i') \sup_{n \in \mathbb{N}} \mathbb{E}[|\delta^n(Z^n)|^2] < \infty \text{ and } \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[|D_i^n \delta^n(Z^n)|^2] < \infty.$$

Thanks to Theorems 9 and 13, assertion (i') is equivalent to

$$(ii') \ Z \in D(\delta), \ \delta(Z) \in \mathbb{D}^{1,2}, \ (\delta^n(Z^n))_{n \in \mathbb{N}} \text{ converges weakly to } \delta(Z) \text{ in } L^2(\Omega, \mathcal{F}, P), \text{ and } (D^n \delta^n(Z^n))_{n \in \mathbb{N}} \text{ converges to } D\delta(Z) \text{ weakly in } L^2(\Omega \times [0, \infty)).$$

Due to the strong convergence of $(Z_{[n \cdot]}^n)_{n \in \mathbb{N}}$ to Z and the weak convergence of $(D_{[n \cdot]}^n \delta^n(Z^n))_{n \in \mathbb{N}}$ to $D(\delta(Z))$, the continuous time duality between Skorokhod integral and Malliavin derivative and its discrete time counterpart in Proposition 14 imply

$$\|\delta^n(Z^n)\|_{L^2(\Omega, \mathcal{F}, P)}^2 = \int_0^\infty \mathbb{E}[Z_{[ns]}^n D_{[ns]}^n \delta^n(Z^n)] ds \rightarrow \int_0^\infty \mathbb{E}[Z_s D_s \delta(Z)] ds = \|\delta(Z)\|_{L^2(\Omega, \mathcal{F}, P)}^2.$$

Hence we obtain the convergence of $(\delta^n(Z^n))_{n \in \mathbb{N}}$ to $\delta(Z)$ in the strong topology, i.e., assertion (ii') is equivalent to assertion (ii). \square

4. STRONG AND WEAK L^2 -APPROXIMATION OF THE ITÔ INTEGRAL AND THE CLARK-OCONE DERIVATIVE

In this section, we first specialize the approximation result for the Skorokhod integral to predictable integrands. In this way, we obtain necessary and sufficient conditions for strong and weak L^2 -convergence of discrete Itô integrals with respect to the noise $(\xi_i^n)_{i \in \mathbb{N}}$ to Itô integrals with respect to the Brownian motion B . Then, we discuss strong and weak L^2 -approximations to the Clark-Ocone derivative, which provides the predictable integral representation of a random variable in $L^2(\Omega, \mathcal{F}, P)$ with respect to the Brownian motion B .

Suppose $Z^n \in L_n^2(\Omega \times \mathbb{N})$ is predictable with respect to $(\mathcal{F}_i^n)_{i \in \mathbb{N}}$, i.e., for every $i \in \mathbb{N}$, Z_i^n is measurable with respect to $\mathcal{F}_{i-1}^n = \sigma(\xi_1^n, \dots, \xi_{i-1}^n)$. Then,

$$\delta^n(Z^n) = \sum_{i=1}^{\infty} Z_i^n \frac{\xi_i^n}{\sqrt{n}} =: \int Z^n dB^n,$$

which means that the discrete Skorokhod integral reduces to the discrete Itô integral. Analogously, the Skorokhod integral $\delta(Z)$ is well-known to coincide with the Itô integral $\int_0^\infty Z_s dB_s$, when $Z \in L^2(\Omega \times [0, \infty))$ is predictable with respect to the augmented Brownian filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, see, e.g. [Janson (1997), Theorem 7.41].

In this case of predictable integrands, the approximation theorem for Skorokhod integrals (Theorem 9) can be improved as follows.

Theorem 21. *Suppose $Z \in L^2(\Omega \times [0, \infty))$ is predictable with respect to the augmented Brownian filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, and, for every $n \in \mathbb{N}$, $Z^n \in L_n^2(\Omega \times \mathbb{N})$ is predictable with respect to $(\mathcal{F}_i^n)_{i \in \mathbb{N}}$. Then, the following are equivalent:*

- (i) $(Z_{[n \cdot]}^n)_{n \in \mathbb{N}}$ converges to Z strongly (resp. weakly) in $L^2(\Omega \times [0, \infty))$.

- (ii) The sequence of discrete Itô integrals $(\int Z^n dB^n)_{n \in \mathbb{N}}$ converges strongly (resp. weakly) in $L^2(\Omega, \mathcal{F}, P)$ to $\int_0^\infty Z_s dB_s$.

Remark 22. We note that, in order to study convergence of Itô integrals (with respect to different filtrations), techniques of convergence in distribution on the Skorokhod space of right-continuous functions with left limits are classically applied. E.g., the results by [Kurtz and Protter (1991)] immediately imply the following result in our setting: Suppose that Z is predictable with respect to the Brownian filtration and its paths are right-continuous with left limits. Moreover, assume that Z^n is predictable with respect to $(\mathcal{F}_i^n)_{i \in \mathbb{N}}$ and $(Z^n_{[1+n(\cdot)]})$ converges to Z uniformly on compacts in probability. Then,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\lfloor n \cdot \rfloor} Z_i^n \frac{1}{\sqrt{n}} \xi_i^n = \int_0^\cdot Z_{s-} dB_s,$$

uniformly on compacts in probability. In contrast, our Theorem 21 provides an L^2 -theory and, in particular, includes the converse implication, namely that convergence of the discrete Itô integrals implies convergence of the integrands.

The proof of Theorem 21 will make use of the following proposition.

Proposition 23. Suppose $g, h \in \mathcal{E}$. Then, strongly in $L^2(\Omega \times [0, \infty))$,

$$\lim_{n \rightarrow \infty} \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor n \cdot \rfloor - 1]}) \check{h}^n(\lceil n \cdot \rceil)) = \exp^{\diamond}(I(g \mathbf{1}_{(0, \cdot]})h(\cdot)).$$

Proof. Recall that the support of h is contained in $[0, M]$ for some $M \in \mathbb{N}$. Hence, we can decompose,

$$\begin{aligned} & \int_0^\infty \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor nt \rfloor - 1]}) \check{h}^n(\lceil nt \rceil)) - \exp^{\diamond}(I(g \mathbf{1}_{(0, t]})h(t)) \right)^2 \right] dt \\ & \leq 2 \int_0^M \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor nt \rfloor]}) \check{h}^n(\lceil nt \rceil)) - \exp^{\diamond}(I(g \mathbf{1}_{(0, t]})h(t)) \right)^2 \right] h(t)^2 dt \\ & \quad + 2 \int_0^\infty \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor nt \rfloor - 1]}) \check{h}^n(\lceil nt \rceil)) \right)^2 \right] |\check{h}^n(\lceil nt \rceil) - h(t)|^2 dt, \end{aligned}$$

since $\lceil nt \rceil - 1 = \lfloor nt \rfloor$ for Lebesgue almost every $t \geq 0$. As, by (4)

$$\sup_{n \in \mathbb{N}, t \in [0, \infty)} \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor nt \rfloor - 1]}) \check{h}^n(\lceil nt \rceil)) \right)^2 \right] \leq \sup_{n \in \mathbb{N}} \exp(\|\check{g}^n(\lceil n \cdot \rceil)\|_{L^2([0, \infty))}^2) < \infty, \quad (23)$$

the second term goes to zero by (7). Moreover, by the boundedness of h , the first one tends to zero by the dominated convergence theorem, since, for every $t \in [0, \infty)$, by Proposition 3,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, \lfloor nt \rfloor]}) \check{h}^n(\lceil nt \rceil)) - \exp^{\diamond}(I(g \mathbf{1}_{(0, t]})h(t)) \right)^2 \right] = 0.$$

□

Proof of Theorem 21. ‘(i) \Rightarrow (ii)’: By the isometry for discrete Itô integrals, we have

$$\mathbb{E} \left[\left(\int Z^n dB^n \right)^2 \right] = \mathbb{E} \left[\left| \sum_{i=1}^{\lfloor n \cdot \rfloor} Z_i^n \frac{1}{\sqrt{n}} \xi_i^n \right|^2 \right] = \int_0^\infty \mathbb{E} [|Z_{[ns]}^n|^2] ds. \quad (24)$$

Hence, if $(Z^n_{[n \cdot]})_{n \in \mathbb{N}}$ converges to Z weakly in $L^2(\Omega, \mathcal{F}, P)$, then the left-hand side in (24) is bounded in $n \in \mathbb{N}$, and so Theorem 9 implies the asserted weak $L^2(\Omega, \mathcal{F}, P)$ convergence of the sequence of discrete Itô integrals to $\int_0^\infty Z_s dB_s$. If $(Z^n_{[n \cdot]})_{n \in \mathbb{N}}$ converges to Z strongly in $L^2(\Omega, \mathcal{F}, P)$, then, by (24) and the continuous time Itô isometry,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\int Z^n dB^n \right)^2 \right] = \int_0^\infty \mathbb{E} [|Z_s|^2] ds = \mathbb{E} \left[\left(\int_0^\infty Z_s dB_s \right)^2 \right],$$

which turns the weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of the sequence of discrete Itô integrals into strong $L^2(\Omega, \mathcal{F}, P)$ -convergence.

‘(ii) \Rightarrow (i)’: We first assume that the sequence of discrete Itô integrals converges weakly in $L^2(\Omega, \mathcal{F}, P)$ to the continuous time Itô integral. By the implication ‘(i) \Rightarrow (ii)’ (which we have already proved) and Proposition 23, we obtain, for every $g, h \in \mathcal{E}$,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, i-1]})) \check{h}^n(i) \frac{1}{\sqrt{n}} \xi_i^n = \int_0^{\infty} \exp^{\diamond}(I(g \mathbf{1}_{(0, s]})) h(s) dB_s \quad (25)$$

strongly in $L^2(\Omega, \mathcal{F}, P)$. As Z^n is predictable and

$$\mathbb{E}[\exp^{\diamond n}(I^n(\check{g}^n)) | \mathcal{F}_{i-1}^n] = \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, i-1]})),$$

we get, for every $g, h \in \mathcal{E}$, by the discrete Itô isometry,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\infty} (S^n Z_i^n)(\check{g}^n) \check{h}^n(i) &= \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{E}[Z_i^n | \mathcal{F}_{i-1}^n] \exp^{\diamond n}(I^n(\check{g}^n)) \check{h}^n(i)] \\ &= \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, i-1]})) \check{h}^n(i)] \\ &= \mathbb{E} \left[\left(\sum_{i=1}^{\infty} Z_i^n \frac{1}{\sqrt{n}} \xi_i^n \right) \left(\sum_{i=1}^{\infty} \exp^{\diamond n}(I^n(\check{g}^n \mathbf{1}_{[1, i-1]})) \check{h}^n(i) \frac{1}{\sqrt{n}} \xi_i^n \right) \right]. \end{aligned}$$

The assumed weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of the sequence of discrete Itô integrals and the strong $L^2(\Omega, \mathcal{F}, P)$ -convergence in (25) now imply

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} (S^n Z_i^n)(\check{g}^n) \check{h}^n(i) = \mathbb{E} \left[\left(\int_0^{\infty} Z_s dB_s \right) \left(\int_0^{\infty} \exp^{\diamond}(I(g \mathbf{1}_{(0, s]})) h(s) dB_s \right) \right].$$

As $(\exp^{\diamond}(I(g \mathbf{1}_{(0, s]})))_{s \in [0, \infty)}$ is a uniformly integrable martingale and Z is predictable, we obtain, by the Itô isometry and the definition of the S -transform,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} (S^n Z_i^n)(\check{g}^n) \check{h}^n(i) = \int_0^{\infty} (SZ_s)(g) h(s) ds, \quad g, h \in \mathcal{E}.$$

We can now apply Theorem 11. As $\int_0^{\infty} \mathbb{E} \left[|Z_{[ns]}^n|^2 \right] ds$ is bounded in n by (24) and by the assumed weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of the discrete Itô integrals, the latter Theorem implies that $(Z_{[n \cdot]}^n)_{n \in \mathbb{N}}$ converges to Z weakly in $L^2(\Omega \times [0, \infty))$. If we instead assume strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of the sequence of the discrete Itô integrals, a straightforward application of the isometries for discrete and continuous-time Itô integrals turns the weak $L^2(\Omega \times [0, \infty))$ -convergence again into strong convergence. \square

We now turn to the Clark-Ocone derivative. Recall that a Brownian motion has the predictable representation property with respect to its natural filtration, i.e., for every $X \in L^2(\Omega, \mathcal{F}, P)$ there is a unique $(\mathcal{F}_t)_{t \in [0, \infty)}$ -predictable process $\nabla X \in L^2(\Omega \times [0, \infty))$ such that

$$X = \mathbb{E}[X] + \int_0^{\infty} \nabla_s X dB_s. \quad (26)$$

We refer to ∇X as the *generalized Clark-Ocone derivative* and recall that $(\nabla_t X)_{t \geq 0}$ is the predictable projection of the Malliavin derivative $(D_t X)_{t \geq 0}$, if $X \in \mathbb{D}^{1,2}$. By Itô's isometry the operator $\nabla : L^2(\Omega, \mathcal{F}, P) \rightarrow L^2(\Omega \times [0, \infty))$ is continuous with norm 1.

Except in the case of binary noise, the discrete time approximation $B^{(n)}$ of the Brownian motion B does not satisfy the discrete time predictable representation property with respect to $(\mathcal{F}_i^n)_{i \in \mathbb{N}}$.

Nonetheless one can consider the discrete time predictable projection of the discretized Malliavin derivative

$$\nabla_i^n X := \mathbb{E}[D_i^n X | \mathcal{F}_{i-1}^n] = \sqrt{n} \mathbb{E}[\xi_i^n X | \mathcal{F}_{i-1}^n], \quad X \in L^2(\Omega, \mathcal{F}, P), \quad i \in \mathbb{N},$$

as discretization of the generalized Clark-Ocone derivative. We refer to $(\nabla_i^n X)_{i \in \mathbb{N}}$ as *discretized Clark-Ocone derivative of X* and note that it has been extensively studied in the context of discretization of backward stochastic differential equations, see, e.g., [Briand et al. (2002), Zhang (2004), Geiss et al. (2012)].

The operator

$$\nabla^n : L^2(\Omega, \mathcal{F}, P) \rightarrow L_n^2(\Omega \times \mathbb{N}), \quad X \mapsto (\nabla_i^n X)_{i \in \mathbb{N}}$$

is continuous with norm one. Indeed, introducing the shorthand notation $\mathbb{E}_{n,i}[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_i^n]$ and noting that the martingale $(\mathbb{E}_{n,i}[X])_{i \in \mathbb{N}}$ is, for fixed $n \in \mathbb{N}$, uniformly integrable, and, thus, converges almost surely to $\mathbb{E}[X | \mathcal{F}^n]$, as i tends to infinity, one gets, by Hölder's and Jensen's inequality,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} \left[(\sqrt{n} \mathbb{E}_{n,i-1} [\xi_i^n X])^2 \right] = \sum_{i=1}^{\infty} \mathbb{E} \left[(\mathbb{E}_{n,i-1} [\xi_i^n (\mathbb{E}_{n,i}[X] - \mathbb{E}_{n,i-1}[X])])^2 \right] \\ & \leq \sum_{i=1}^{\infty} \mathbb{E} \left[\mathbb{E}_{n,i-1} [(\xi_i^n)^2] \mathbb{E}_{n,i-1} [(\mathbb{E}_{n,i}[X] - \mathbb{E}_{n,i-1}[X])^2] \right] \\ & = \mathbb{E} \left[\sum_{i=1}^{\infty} ((\mathbb{E}_{n,i}[X])^2 - (\mathbb{E}_{n,i-1}[X])^2) \right] = \mathbb{E} \left[(\mathbb{E}[X | \mathcal{F}^n])^2 \right] - \mathbb{E}[X]^2 \\ & \leq \mathbb{E}[(X)^2] - \mathbb{E}[X]^2. \end{aligned}$$

We now denote by

$$\mathcal{P}^n := \left\{ a + \int Z^n dB^n; a \in \mathbb{R}, Z^n \in L_n^2(\Omega \times \mathbb{N}) \text{ predictable} \right\}$$

the closed subspace in $L^2(\Omega, \mathcal{F}, P)$, which admits a discrete time predictable integral representation. Note that, for every $X \in L^2(\Omega, \mathcal{F}, P)$, $a \in \mathbb{R}$, and $(\mathcal{F}_i^n)_{i \in \mathbb{N}}$ -predictable $Z^n \in L_n^2(\Omega \times \mathbb{N})$, by the discrete Itô isometry,

$$\begin{aligned} \mathbb{E} \left[X \left(a + \int Z^n dB^n \right) \right] &= a \mathbb{E}[X] + \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} \mathbb{E}[X \xi_i^n \mathbb{E}[Z_i^n | \mathcal{F}_{i-1}^n]] \\ &= a \mathbb{E}[X] + \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} [Z_i^n \sqrt{n} \mathbb{E}[X \xi_i^n | \mathcal{F}_{i-1}^n]] = \mathbb{E} \left[\left(\mathbb{E}[X] + \int \nabla^n X dB^n \right) \left(a + \int Z^n dB^n \right) \right]. \end{aligned}$$

Hence,

$$\pi_{\mathcal{P}^n} X = \mathbb{E}[X] + \int \nabla^n X dB^n, \quad (27)$$

where, for any closed subspace \mathcal{A} in $L^2(\Omega, \mathcal{F}, P)$, $\pi_{\mathcal{A}}$ denotes the orthogonal projection on \mathcal{A} . Our first approximation result for the Clark-Ocone derivative now reads as follows:

Theorem 24. *Suppose $(X^n)_{n \in \mathbb{N}}$ is a sequence in $L^2(\Omega, \mathcal{F}, P)$ and $X \in L^2(\Omega, \mathcal{F}, P)$. Then, the following are equivalent, as n tends to infinity:*

- (i) $(\pi_{\mathcal{P}^n} X^n - \mathbb{E}[X^n])_{n \in \mathbb{N}}$ converges to $X - \mathbb{E}[X]$ strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$.
- (ii) $(\nabla_{[n \cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to ∇X strongly (weakly) in $L^2(\Omega \times [0, \infty))$.

A sufficient condition for (i), (ii) is that $(X^n)_{n \in \mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$.

Proof. Recall that by (26) and (27)

$$X - \mathbb{E}[X] = \int_0^\infty \nabla X_s dB_s,$$

$$\pi_{\mathcal{P}^n} X^n - \mathbb{E}[X^n] = \int \nabla^n X^n dB^n.$$

Hence, Theorem 21 provides the equivalence of (i) and (ii). As, for every $g \in \mathcal{E}$, $\exp^{\diamond n}(I^n(\check{g}^n)) \in \mathcal{P}^n$ by (5), the sufficient condition is a consequence of the following lemma. \square

Lemma 25. *Suppose that \mathcal{A}^n , $n \in \mathbb{N}$, are closed subspaces of $L^2(\Omega, \mathcal{F}, P)$ such that for every $n \in \mathbb{N}$,*

$$\{\exp^{\diamond n}(I^n(\check{g}^n)), g \in \mathcal{E}\} \subset \mathcal{A}^n.$$

Then, strong (weak) $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(X^n)_{n \in \mathbb{N}}$ to X implies that $(\pi_{\mathcal{A}^n} X^n)_{n \in \mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$ as well.

Proof. As, for every $g \in \mathcal{E}$,

$$\mathbb{E}[(\pi_{\mathcal{A}^n} X^n) \exp^{\diamond n}(I^n(\check{g}^n))] = \mathbb{E}[X^n \pi_{\mathcal{A}^n}(\exp^{\diamond n}(I^n(\check{g}^n)))] = \mathbb{E}[X^n \exp^{\diamond n}(I^n(\check{g}^n))],$$

we obtain that $(S^n X^n)(\check{g}^n) = (S^n(\pi_{\mathcal{A}^n} X^n))(\check{g}^n)$. In the case of weak convergence, Theorem 1 now immediately applies, because

$$\mathbb{E}[(\pi_{\mathcal{A}^n} X^n)^2] \leq \mathbb{E}[X^n^2].$$

In the case of strong convergence, we also make use of Theorem 1, and note that by the already established weak convergence of $(\pi_{\mathcal{A}^n} X^n)_{n \in \mathbb{N}}$ and Hölder's inequality,

$$\mathbb{E}[(\pi_{\mathcal{A}^n} X^n)^2] = \mathbb{E}[X(\pi_{\mathcal{A}^n} X^n)] + \mathbb{E}[(X^n - X)(\pi_{\mathcal{A}^n} X^n)] \rightarrow \mathbb{E}[X^2], \quad n \rightarrow \infty.$$

\square

We shall finally discuss an alternative approximation of the generalized Clark-Ocone derivative, which involves orthogonal projections on appropriate finite-dimensional subspaces. To this end, we denote by \mathcal{H}^n the strong closure in $L^2(\Omega, \mathcal{F}, P)$ of the linear span of

$$\Xi^n := \left\{ \Xi_A^n := \prod_{i \in A} \xi_i^n, \quad A \subseteq \mathbb{N}, |A| < \infty \right\},$$

and emphasize that $\mathcal{H}^n = L^2(\Omega, \mathcal{F}^n, P)$, if and only if the noise distribution of ξ is binary. As Ξ^n consists of an orthonormal basis of \mathcal{H}^n , every $X^n \in \mathcal{H}^n$ has a unique expansion in terms of this Hilbert space basis, which is called the *Walsh decomposition* of X^n ,

$$X^n = \sum_{|A| < \infty} X_A^n \Xi_A^n, \tag{28}$$

where $X_A^n = \mathbb{E}[X^n \Xi_A^n]$ satisfies $\sum_{|A| < \infty} (X_A^n)^2 < \infty$. The expectation and $L^2(\Omega, \mathcal{F}, P)$ -inner product can be computed in terms of the Walsh decomposition via $\mathbb{E}[X^n] = X_\emptyset^n$ and

$$\mathbb{E}[X^n Y^n] = \sum_{|A| < \infty} X_A^n Y_A^n, \quad X^n, Y^n \in \mathcal{H}^n, \tag{29}$$

cp. [Holden et al. (1992)]. A direct computation shows that the Walsh decomposition of a discrete Wick exponential is given by

$$\exp^{\diamond n}(I^n(f^n)) = \sum_{|A| < \infty} \left(n^{-|A|/2} \prod_{i \in A} f^n(i) \right) \Xi_A^n. \tag{30}$$

In view of the Möbius inversion formula [Aigner (2007), Theorem 5.5], we obtain, for every finite subset B of \mathbb{N} ,

$$\Xi_B^n = n^{|B|/2} \sum_{C \subseteq B} (-1)^{|B|-|C|} \exp^{\diamond n}(I^n(\mathbf{1}_C)). \tag{31}$$

Hence, the set $\{\exp^{\diamond n}(I^n(\check{g}^n)), g \in \mathcal{E}\}$ is total in \mathcal{H}^n .

We now consider the finite-dimensional subspaces

$$\mathcal{H}_i^n := \text{span}\{\Xi_A^n, A \subset \{1, \dots, i\}\},$$

and introduce, as a second approximation of the generalized Clark-Ocone derivative, the operator

$$\bar{\nabla}^n : L^2(\Omega, \mathcal{F}, P) \rightarrow L_n^2(\Omega \times \mathbb{N}), \quad X \mapsto (\pi_{\mathcal{H}_{i-1}^n}(\nabla_i^n X))_{i \in \mathbb{N}}.$$

Notice that

$$\bar{\nabla}_i^n X = \sqrt{n} \pi_{\mathcal{H}_{i-1}^n}(\xi_i^n X),$$

if $\xi_i^n X \in L^2(\Omega, \mathcal{F}, P)$.

We are now going to show the following variant of Theorem 24.

Theorem 26. *Suppose $(X^n)_{n \in \mathbb{N}}$ is a sequence in $L^2(\Omega, \mathcal{F}, P)$ and $X \in L^2(\Omega, \mathcal{F}, P)$. Then, the following are equivalent, as n tends to infinity:*

- (i) $(\pi_{\mathcal{H}^n} X^n - \mathbb{E}[X^n])_{n \in \mathbb{N}}$ converges to $X - \mathbb{E}[X]$ strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$.
- (ii) $(\bar{\nabla}_{[n, \cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to ∇X strongly (weakly) in $L^2(\Omega \times [0, \infty))$.

A sufficient condition for (i), (ii) is that $(X^n)_{n \in \mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$.

The proof is based on the simple observation that $\mathcal{H}^n \subset \mathcal{P}^n$, i.e., for every $X^n \in \mathcal{H}^n$,

$$X^n = \mathbb{E}[X^n] + \sum_{i=1}^{\infty} \nabla_i^n X^n \frac{1}{\sqrt{n}} \xi_i^n. \quad (32)$$

In order to show this, we recall that $\{\exp^{\diamond n}(I^n(\check{g}^n)), g \in \mathcal{E}\}$ is total in \mathcal{H}^n . Thus, by continuity of the discretized Clark-Ocone derivative and by the discrete Itô isometry, it suffices to show (32) in the case $X^n = \exp^{\diamond n}(I^n(\check{f}^n))$ for $f \in \mathcal{E}$. A direct computation shows,

$$\nabla_i^n \exp^{\diamond n}(I^n(f^n)) = f^n(i) \exp^{\diamond n}(I^n(f^n \mathbf{1}_{[1, i-1]})), \quad (33)$$

which in view of (5) completes the proof of (32).

Proof of Theorem 26. We first note that, for every $X \in L^2(\Omega, \mathcal{F}, P)$,

$$\mathbb{E}[\pi_{\mathcal{H}^n} X] = \mathbb{E}[X], \quad (34)$$

$$\bar{\nabla}_i^n X = \nabla_i^n(\pi_{\mathcal{H}^n} X). \quad (35)$$

Indeed, as

$$\pi_{\mathcal{H}^n} X = \mathbb{E}[X] + \sum_{1 \leq |A| < \infty} \mathbb{E}[X \Xi_A^n] \Xi_A^n,$$

Eq. (34) is obvious. In order to prove (35), we recall first that $\nabla_i^n(\pi_{\mathcal{H}^n} X) \in \mathcal{H}_{i-1}^n$ (by (33) and continuity of the discretized Clark-Ocone derivative) and then note that, for every $A \subset \{1, \dots, i-1\}$,

$$\begin{aligned} \mathbb{E}[\Xi_A^n \mathbb{E}[\xi_i^n X | \mathcal{F}_{i-1}^n]] &= \mathbb{E}[\Xi_{A \cup \{i\}}^n X] = \mathbb{E}[\Xi_{A \cup \{i\}}^n \pi_{\mathcal{H}^n}(X)] \\ &= \mathbb{E}[\Xi_A^n \mathbb{E}[\xi_i^n \pi_{\mathcal{H}^n}(X) | \mathcal{F}_{i-1}^n]] = \mathbb{E}\left[\Xi_A^n \frac{1}{\sqrt{n}} \nabla_i^n(\pi_{\mathcal{H}^n} X)\right]. \end{aligned}$$

In particular, by (32), (34), and (35)

$$\pi_{\mathcal{H}^n} X = \mathbb{E}[X] + \int \bar{\nabla}^n X dB^n, \quad (36)$$

which is the analogue of (27). The proof of Theorem 24 can now be repeated verbatim with \mathcal{P}^n replaced by \mathcal{H}^n . \square

We close this section with two remarks.

Remark 27. In view of Lemma 25 and the inclusion $\mathcal{H}^n \subset \mathcal{P}^n$ we observe that, for any sequence $(X^n)_{n \in \mathbb{N}}$ in $L^2(\Omega, \mathcal{F}, P)$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P) \\ \Rightarrow & \lim_{n \rightarrow \infty} \pi_{\mathcal{P}^n} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P) \\ \Rightarrow & \lim_{n \rightarrow \infty} \pi_{\mathcal{H}^n} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P). \end{aligned}$$

In particular, by Theorems 24 and 26, if the sequence of discretized Clark-Ocone derivatives $(\nabla_{[n \cdot]}^n X^n)_{n \in \mathbb{N}}$ converges to ∇X strongly (weakly) in $L^2(\Omega \times [0, \infty))$, then so does the sequence of modified discretized Clark-Ocone derivatives $(\bar{\nabla}_{[n \cdot]}^n X^n)_{n \in \mathbb{N}}$.

Remark 28. The following result can be derived from [Briand et al. (2002), Theorem 5 and the examples in Section 5] under the additional assumption that $\mathbb{E}[|\xi|^{2+\epsilon}] < \infty$ for some $\epsilon > 0$ and on a finite time horizon: Strong convergence of $(X^n)_{n \in \mathbb{N}}$ to X in $L^2(\Omega, \mathcal{F}, P)$ implies convergence of the sequence of discretized Clark-Ocone derivatives as stated in (ii) of Theorem 24. Our Theorem 26 additionally shows that the conditional expectations $\mathbb{E}[\cdot | \mathcal{F}_{i-1}^n]$ in the definition of the discretized Clark-Ocone derivative can be replaced by the projection on the finite dimensional subspace \mathcal{H}_i^n , i.e., if $(X^n)_{n \in \mathbb{N}}$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$, then

$$\left(\sqrt{n} \pi_{\mathcal{H}_{[nt]-1}^n} (\xi_{[nt]}^n (\tau_n(X^n))) \right)_{t \in [0, \infty)} \rightarrow \nabla X$$

strongly in $L^2(\Omega \times [0, \infty))$, where τ_n denotes the truncation at $\pm n$.

We also note that, in view of (36),

$$\bar{\nabla}_i X = \frac{(\pi_{\mathcal{H}_i^n} X) - (\pi_{\mathcal{H}_{i-1}^n} X)}{B_i^n - B_{i-1}^n}$$

can be rewritten as difference operator (where we apply the convention $\frac{\xi_i^n}{\xi_i^n} = 1$ when ξ_i^n vanishes). This representation shows the close relation to the weak $L^2(\Omega \times [0, \infty))$ -approximation result for the generalized Clark-Ocone derivative which is derived in [Leão and Ohashi (2013), Corollary 4.1], but for the case of symmetric binary noise only.

5. STRONG L^2 -APPROXIMATION OF THE CHAOS DECOMPOSITION

In this section, we apply Theorem 1 in order to characterize strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of a sequence (X^n) (where X^n can be represented via multiple Wiener integrals with respect to the discrete time noise $(\xi_i^n)_{i \in \mathbb{N}}$) via convergence of the coefficient functions of such a discrete chaos decomposition.

Recall first, that every $X \in L^2(\Omega, \mathcal{F}, P)$ has a unique Wiener chaos decomposition in terms of multiple Wiener integrals

$$X = \sum_{k=0}^{\infty} I^k(f_X^k), \tag{37}$$

where $f_X^k \in \widetilde{L}^2([0, \infty)^k)$, see e.g. [Nualart (2006), Theorem 1.1.2]. Here, we denote by $L^2([0, \infty)^k)$ the Hilbert space of square-integrable functions with respect to the k -dimensional Lebesgue measure and by $\widetilde{L}^2([0, \infty)^k)$ the subspace of functions in $L^2([0, \infty)^k)$ which are symmetric in the k variables. We apply the standard convention $\widetilde{L}^2([0, \infty)^0) = L^2([0, \infty)^0) = \mathbb{R}$, $I^0(f^0) = f^0$, and recall that, for $k \geq 1$ and $f^k \in \widetilde{L}^2([0, \infty)^k)$, the multiple Wiener integral can be defined as iterated Itô integral:

$$I^k(f^k) = k! \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} f^k(t_1, \dots, t_k) dB_{t_1} \cdots dB_{t_{k-1}} dB_{t_k}.$$

The Itô isometry therefore immediately implies the following well-known Wiener-Itô isometry for multiple Wiener integrals,

$$\mathbb{E}[I^k(f^k) I^{k'}(g^{k'})] = \delta_{k,k'} k! \langle f^k, g^{k'} \rangle_{L^2([0,\infty)^k)} \quad (38)$$

for functions $f^k \in \widetilde{L}^2([0,\infty)^k)$ and $g^{k'} \in \widetilde{L}^2([0,\infty)^{k'})$.

The main theorem of this section now reads as follows:

Theorem 29 (Wiener chaos limit theorem). *Suppose $(X^n)_{n \in \mathbb{N}}$ is a sequence in $L^2(\Omega, \mathcal{F}, P)$. Then the following assertions are equivalent as n tends to infinity:*

- (i) *The sequence $(\pi_{\mathcal{H}^n} X^n)$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$.*
- (ii) *For every $k \in \mathbb{N}_0$, the sequence $(\widehat{f_{X^n}^{n,k}})_{n \in \mathbb{N}}$, defined via*

$$\widehat{f_{X^n}^{n,k}}(u_1, \dots, u_k) := \begin{cases} \mathbb{E} \left[X^n \frac{n^{k/2}}{k!} \Xi_{\{[nu_1], \dots, [nu_k]\}}^n \right], & |\{[nu_1], \dots, [nu_k]\} \cap \mathbb{N}| = k, \\ 0, & \text{otherwise,} \end{cases} \quad (39)$$

is strongly convergent in $L^2([0,\infty)^k)$ and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=m}^{\infty} k! \|\widehat{f_{X^n}^{n,k}}\|_{L^2([0,\infty)^k)}^2 = 0. \quad (40)$$

In this case, the limit X of $(\pi_{\mathcal{H}^n} X^n)_{n \in \mathbb{N}}$ has the Wiener chaos decomposition $X = \sum_{k=0}^{\infty} I^k(f_X^k)$

with $f_X^k = \lim_{n \rightarrow \infty} \widehat{f_{X^n}^{n,k}}$ in $L^2([0,\infty)^k)$.

We recall that, by Remark 27, the strong $L^2(\Omega, \mathcal{F}, P)$ -convergence of (X^n) to X is a sufficient condition for the strong approximation of the chaos coefficients of X as stated in (ii) of the above theorem.

Before proving Theorem 29, we briefly discuss this result. To this end, we first recall the relation between Walsh decomposition and discrete chaos decomposition. The discrete multiple Wiener integrals are defined analogously to the continuous setting, see e.g. [Privault (2009), Section 1.3]. For all $k, n \in \mathbb{N}$ we consider the Hilbert space

$$L_n^2(\mathbb{N}^k) := \left\{ f^{n,k} : \mathbb{N}^k \rightarrow \mathbb{R} : \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k} \left(f^{n,k}(i_1, \dots, i_k) \right)^2 < \infty \right\}$$

endowed with the inner product

$$\langle f^{n,k}, g^{n,k} \rangle_{L_n^2(\mathbb{N}^k)} := n^{-k} \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k} f^{n,k}(i_1, \dots, i_k) g^{n,k}(i_1, \dots, i_k).$$

The closed subspace of symmetric functions in $L_n^2(\mathbb{N}^k)$ which vanish on the diagonal part

$$\partial_k := \left\{ (i_1, \dots, i_k) \in \mathbb{N}^k : |\{i_1, \dots, i_k\}| < k \right\}$$

is denoted by $\widetilde{L}_n^2(\mathbb{N}^k)$.

Then, for $k \in \mathbb{N}$, the discrete multiple Wiener integral of $f^{n,k} \in \widetilde{L}_n^2(\mathbb{N}^k)$ with respect to the random walk B^n is defined as

$$I^{n,k}(f^{n,k}) := n^{-k/2} k! \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k, i_1 < \dots < i_k} f^{n,k}(i_1, \dots, i_k) \Xi_{\{i_1, \dots, i_k\}}^n.$$

We notice that $I^{n,k}$ is linear on $\widetilde{L}_n^2(\mathbb{N}^k)$ and fulfills $\mathbb{E}[I^{n,k}(f^{n,k})] = 0$ as well as the isometry

$$\mathbb{E}[I^{n,k}(f^{n,k}) I^{n,k'}(g^{n,k'})] = \delta_{k,k'} k! \langle f^{n,k}, g^{n,k'} \rangle_{L_n^2(\mathbb{N}^k)} \quad (41)$$

for $f^{n,k} \in \widetilde{L}_n^2(\mathbb{N}^k)$, $g^{n,k'} \in \widetilde{L}_n^2(\mathbb{N}^{k'})$ and possibly different orders $k, k' \in \mathbb{N}$. As in the continuous time setting, we apply the convention that $I^{n,0}$ is the identity on $\widetilde{L}_n^2(\mathbb{N}^0) := \mathbb{R}$, and refer to [Privault (2009), Section 1.3] for further properties of such discrete multiple Wiener integrals. We now fix $X^n \in \mathcal{H}^n$. In view of the Walsh decomposition $X^n = \sum_{|A| < \infty} \mathbb{E}[X^n \Xi_A^n] \Xi_A^n$, we observe that the discrete analog of the Wiener chaos decomposition

$$X^n = \sum_{k=0}^{\infty} n^{-k/2} k! \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k, i_1 < \dots < i_k} \frac{n^{k/2}}{k!} X^n_{\{i_1, \dots, i_k\}} \Xi_{\{i_1, \dots, i_k\}}^n = \sum_{k=0}^{\infty} I^{n,k}(f_{X^n}^{n,k}), \quad (42)$$

holds for the integrands $f_{X^n}^{n,k} \in \widetilde{L}_n^2(\mathbb{N}^k)$ given by

$$f_{X^n}^{n,k}(i_1, \dots, i_k) := \begin{cases} \mathbb{E} \left[\frac{n^{k/2}}{k!} X^n \Xi_{\{i_1, \dots, i_k\}}^n \right], & |\{i_1, \dots, i_k\} \cap \mathbb{N}| = k \\ 0, & \text{otherwise.} \end{cases} \quad (43)$$

Hence, this discrete analog of the Wiener chaos decomposition (37) for random variables in \mathcal{H}^n is just a reformulation of the Walsh decomposition (28).

Given a general element $f^{n,k} \in \widetilde{L}_n^2(\mathbb{N}^k)$ we define its embedding into simple continuous time functions in k variables as

$$\begin{aligned} \widehat{f^{n,k}}(u_1, \dots, u_k) &:= f^{n,k}(\lceil nu_1 \rceil, \dots, \lceil nu_k \rceil) \\ &= \sum_{i_1, \dots, i_k=1}^{\infty} f^{n,k}(i_1, \dots, i_k) \mathbf{1}_{(\frac{i_1-1}{n}, \frac{i_1}{n}] \times \dots \times (\frac{i_k-1}{n}, \frac{i_k}{n}]}(u_1, \dots, u_k), \end{aligned} \quad (44)$$

which is consistent with the notation already applied in (39) and (43). Here and in what follows, we apply the convention that $f^{n,k}$ vanishes when one of its arguments is set to zero.

We can now rephrase Theorem 29 in the following way:

The sequence (X^n) , with $X^n \in \mathcal{H}^n$, converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$, if and only if, for all orders $k \in \mathbb{N}_0$, the sequence of coefficient functions of the discrete chaos decomposition of X^n converge (after the natural embedding into continuous time) to the coefficient functions of the Wiener chaos of X strongly in $L^2([0, \infty)^k)$ and the tail condition (40) is satisfied.

Remark 30. *Convergence of discrete multiple Wiener integrals to continuous multiple Wiener integrals was studied in [Surgailis (1982)] as a main tool for proving noncentral limit theorems. The results in Section 4 of the latter reference imply that, for every $k \in \mathbb{N}_0$, the sequence of discrete multiple Wiener integrals $(I^{n,k}(f^{n,k}))_{n \in \mathbb{N}}$ converges in distribution to the multiple Wiener integral $I^k(f^k)$, if $(\widehat{f^{n,k}})_{n \in \mathbb{N}}$ converges to f^k strongly in $L^2([0, \infty)^k)$. Our result lifts this convergence in distribution to strong $L^2(\Omega, \mathcal{F}, P)$ -convergence and adds the converse:*

$$L^2(\Omega, \mathcal{F}, P)\text{-}\lim_{n \rightarrow \infty} I^{n,k}(f^{n,k}) = I^k(f^k) \Leftrightarrow L^2([0, \infty)^k)\text{-}\lim_{n \rightarrow \infty} \widehat{f^{n,k}} = f^k.$$

We note that the $L^2(\Omega, \mathcal{F}, P)$ -convergence of the sequence $(I^{n,k}(f^{n,k}))$ even implies convergence in $L^p(\Omega, \mathcal{F}, P)$ for $p > 2$, if $\mathbb{E}[|\xi|^r] < \infty$ for some $r > p$. Indeed, in this case, the sequence $(|I^{n,k}(f^{n,k})|^p)$ is uniformly integrable by the hypercontractivity inequality of [Krakowiak and Szulga (1986)] in the variant of [Bai and Taqqu (2014), Proposition 5.2].

The following elementary corollary of Theorem 29 generalizes Proposition 3. It makes use of the fact that the chaos decompositions of (discrete) Wick exponentials are given, for all $f \in L^2([0, \infty))$, $f^n \in L_n^2(\mathbb{N})$, by

$$\exp^\diamond(I(f)) = \sum_{k=0}^{\infty} I^k\left(\frac{1}{k!} f^{\otimes k}\right), \quad \exp^{\diamond n}(I^n(f^n)) = \sum_{k=0}^{\infty} I^{n,k}\left(\frac{1}{k!} (f^n)^{\otimes k} \mathbf{1}_{\partial_k^c}\right). \quad (45)$$

For a proof of the continuous case see e.g. [Janson (1997), Theorem 3.21, Theorem 7.26]. The statement of the discrete case is a direct consequence of (30).

Corollary 31. Suppose $f \in L^2([0, \infty))$ and (f^n) is a sequence with $f^n \in L_n^2(\mathbb{N})$ for every $n \in \mathbb{N}$. Then, as n tends to infinity (in the sense of strong convergence),

$$\begin{aligned} \widehat{f^n} \rightarrow f \text{ in } L^2([0, \infty)) &\Leftrightarrow I^n(f^n) \rightarrow I(f) \text{ in } L^2(\Omega, \mathcal{F}, P) \\ &\Leftrightarrow \exp^{\diamond n}(I^n(f^n)) \rightarrow \exp^{\diamond}(I(f)) \text{ in } L^2(\Omega, \mathcal{F}, P). \end{aligned}$$

Proof. In view of Theorem 29 and (45), we only have to show that $\widehat{f^n} \rightarrow f$ strongly in $L^2([0, \infty))$ implies that $(\widehat{f^n})^{\otimes k} \mathbf{1}_{\partial_k^c} \rightarrow f^{\otimes k}$ strongly in $L^2([0, \infty)^k)$, for every $k \geq 2$. This is a consequence of the following lemma. \square

Lemma 32. (i) Fix $k \in \mathbb{N}_0$. Suppose $(f^{n,k})_{n \in \mathbb{N}}$ is a sequence such that $f^{n,k} \in L_n^2(\mathbb{N}^k)$ for every $n \in \mathbb{N}$ and $(\widehat{f^{n,k}})$ converges to some f^k strongly in $L^2([0, \infty)^k)$. Then, the sequence $(\widehat{f^{n,k}} \mathbf{1}_{\partial_k^c})$ converges to f^k strongly in $L^2([0, \infty)^k)$ as well.

(ii) Suppose $(f^n)_{n \in \mathbb{N}}$ is a sequence such that $f^n \in L_n^2(\mathbb{N})$ for every $n \in \mathbb{N}$ and $(\widehat{f^n})$ converges to some f strongly in $L^2([0, \infty))$. Then, for every $k \geq 2$, the sequences $((\widehat{f^n})^{\otimes k})$ and $((\widehat{f^n})^{\otimes k} \mathbf{1}_{\partial_k^c})$ converge to $f^{\otimes k}$ strongly in $L^2([0, \infty)^k)$.

Proof. (i) We decompose,

$$\|\widehat{f^{n,k}} \mathbf{1}_{\partial_k^c} - f^k\|_{L^2([0, \infty)^k)} \leq \|\widehat{f^{n,k}} \mathbf{1}_{\partial_k^c} - \widehat{f^{n,k}}\|_{L^2([0, \infty)^k)} + \|\widehat{f^{n,k}} - f^k\|_{L^2([0, \infty)^k)}.$$

The second term goes to zero by assumption. The first one equals

$$\left(\int_{[0, \infty)^k} |f^{n,k}(\lceil nu_1 \rceil, \dots, \lceil nu_k \rceil)|^2 \mathbf{1}_{\{|\{\lceil nu_1 \rceil, \dots, \lceil nu_k \rceil\}| < k\}} \right)^{1/2}.$$

The sequence of integrands tends to 0 almost everywhere, because

$$\lim_{n \rightarrow \infty} \mathbf{1}_{\{|\{\lceil nu_1 \rceil, \dots, \lceil nu_k \rceil\}| < k\}} = \mathbf{1}_{\{u_l = u_p, \text{ for some } l \neq p\}}.$$

Moreover, the sequence of integrands inherits uniform integrability from the $L^2([0, \infty)^k)$ -convergent series $(\widehat{f^{n,k}})$. Therefore, the first term goes to zero by interchanging limit and integration.

(ii) As tensor powers commute with discretization and embedding, i.e.

$$(\check{g}^n)^{\otimes k} = ((g)^{\check{\otimes k}})^n, \quad \widehat{h^n}^{\otimes k} = \widehat{(h^n)^{\otimes k}} \quad (46)$$

for all $k \in \mathbb{N}$, $g \in \mathcal{E}$, $h^n \in L_n^2(\mathbb{N})$, and as the tensor product is continuous, we observe inductively that $(\widehat{f^n})^{\otimes k} \rightarrow f^{\otimes k}$ strongly in $L^2([0, \infty)^k)$. Then, for the second sequence, part (i) applies. \square

We now start to prepare the proof of Theorem 29.

Proposition 33. Let $k \in \mathbb{N}_0$. Then, for all $g \in \mathcal{E}$ and sequences $(f^{n,k})_{n \in \mathbb{N}}$ such that $f^{n,k} \in \widehat{L_n^2}(\mathbb{N}^k)$ and $\sup_{n \in \mathbb{N}} \|f^{n,k}\|_{L_n^2(\mathbb{N})} < \infty$,

$$\lim_{n \rightarrow \infty} \left| (S^n I^{n,k}(f^{n,k}))(\check{g}^n) - (SI^k(\widehat{f^{n,k}}))(g) \right| = 0.$$

Proof. First note that, by (41), (45), and as $f^{n,k}$ vanishes on the diagonal ∂_k ,

$$\begin{aligned} (S^n I^{n,k}(f^{n,k}))(\check{g}^n) &= \mathbb{E} \left[I^{n,k}(f^{n,k}) \exp^{\diamond n}(I^n(\check{g}^n)) \right] = \langle f^{n,k}, (\check{g}^n)^{\otimes k} \rangle_{L_n^2(\mathbb{N}^k)} \\ &= \int_{[0, \infty)^k} \widehat{f^{n,k}}(x_1, \dots, x_k) (\widehat{\check{g}^n})^{\otimes k}(x_1, \dots, x_k) dx_1 \cdots dx_k. \end{aligned}$$

Analogously, making use of the Wiener-Itô isometry for the continuous chaos decomposition (38) instead of (41), we get

$$(SI^k(\widehat{f^{n,k}}))(g) = \int_{[0, \infty)^k} \widehat{f^{n,k}}(x_1, \dots, x_k) g^{\otimes k}(x_1, \dots, x_k) dx_1 \cdots dx_k.$$

Hence, by the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned} & \left| (S^n I^{n,k}(f^{n,k}))(\check{g}^n) - (SI^k(\widehat{f^{n,k}}))(g) \right| \\ &= \left| \int_{[0,\infty)^k} \widehat{f^{n,k}}(x_1, \dots, x_k) \left((\widehat{\check{g}^n})^{\otimes k} - g^{\otimes k} \right) (x_1, \dots, x_k) dx_1 \cdots dx_k \right| \\ &\leq \left(\sup_{m \in \mathbb{N}} \|f^{m,k}\|_{L_n^2(\mathbb{N})} \right)^{1/2} \|g^{\otimes k} - (\widehat{\check{g}^n})^{\otimes k}\|_{L^2([0,\infty)^k)}, \end{aligned}$$

which tends to zero for $n \rightarrow \infty$ by Lemma 32. \square

Corollary 34. *Suppose $g \in \mathcal{E}$. Then, for every $k \in \mathbb{N}$,*

$$I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c}) \rightarrow I^k(g^{\otimes k})$$

strongly in $L^2(\Omega, \mathcal{F}, P)$.

Proof. We check item (ii) in Theorem 1. To this end, we decompose, for every $g, h \in \mathcal{E}$,

$$\begin{aligned} & \left| (S^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c}))(\check{h}^n) - (SI^k(g^{\otimes k}))(h) \right| \\ &\leq \left| (S^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c}))(\check{h}^n) - (SI^k((\widehat{\check{g}^n})^{\otimes k} \mathbf{1}_{\partial_k^c}))(\check{h}^n) \right| \\ &\quad + \left| (SI^k((\widehat{\check{g}^n})^{\otimes k} \mathbf{1}_{\partial_k^c}))(\check{h}^n) - (SI^k(g^{\otimes k}))(h) \right|. \end{aligned}$$

The first term on the righthand side tends to zero by Proposition 33. The second one equals, by the isometry for multiple Wiener integrals,

$$\int_{[0,\infty)^k} h^{\otimes k}(x) \left((\widehat{\check{g}^n})^{\otimes k} \mathbf{1}_{\partial_k^c} - g^{\otimes k} \right) (x) dx$$

and goes to zero by Lemma 32. Consequently,

$$\lim_{n \rightarrow \infty} (S^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c}))(\check{h}^n) = (SI^k(g^{\otimes k}))(h)$$

for all $k \in \mathbb{N}_0$ and $g, h \in \mathcal{E}$. For $h = g$, this implies $\mathbb{E}[I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})^2] \rightarrow \mathbb{E}[I^k(g^{\otimes k})^2]$ by the orthogonality of (discrete) multiple Wiener integrals of different orders. Thus, Theorem 1 applies. \square

We are now in the position to give the proof of Theorem 29.

Proof of Theorem 29. ‘(i) \Rightarrow (ii)’: We denote the limit of $(\pi_{\mathcal{H}^n} X^n)_{n \in \mathbb{N}}$ in $L^2(\Omega, \mathcal{F}, P)$ by X and recall that

$$\pi_{\mathcal{H}^n} X^n = \sum_{|A| < \infty} \mathbb{E}[X^n \Xi_A^n] \Xi_A^n = \sum_{k=0}^{\infty} I^{n,k}(f_{X^n}^{n,k}),$$

with $f_{X^n}^{n,k}$ as defined in (43). Throughout the proof we omit the subscripts from the coefficients of the chaos decompositions and write $\pi_{\mathcal{H}^n} X^n = \sum_{k=0}^{\infty} I^{n,k}(f^{n,k})$ and $X = \sum_{k=0}^{\infty} I^k(f^k)$. Thanks to Corollary 34 and the orthogonality of (discrete) multiple Wiener integrals of different orders, we obtain, for every $k \in \mathbb{N}_0$,

$$(S^n I^{n,k}(f^{n,k}))(\check{g}^n) = \frac{1}{k!} \mathbb{E}[\pi_{\mathcal{H}^n}(X^n) I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \rightarrow \frac{1}{k!} \mathbb{E}[X I^k(g^{\otimes k})] = (SI^k(f^k))(g).$$

The estimate $\sup_{n \in \mathbb{N}} \mathbb{E}[(I^{n,k}(f^{n,k}))^2] \leq \sup_{n \in \mathbb{N}} \mathbb{E}[(\pi_{\mathcal{H}^n} X^n)^2] < \infty$ now yields, in view of Theorem 1, weak $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(I^{n,k}(f^{n,k}))_{n \in \mathbb{N}}$ towards $I^k(f^k)$. As $\pi_{\mathcal{H}^n} X^n \rightarrow X$ strongly in $L^2(\Omega, \mathcal{F}, P)$, we thus obtain

$$\mathbb{E}[(I^{n,k}(f^{n,k}))^2] = \mathbb{E}[I^{n,k}(f^{n,k}) \pi_{\mathcal{H}^n} X^n] \rightarrow \mathbb{E}[I^k(f^k) X] = \mathbb{E}[(I^k(f^k))^2]. \quad (47)$$

Hence, $I^{n,k}(f^{n,k}) \rightarrow I^k(f^k)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ for all $k \in \mathbb{N}_0$ by Theorem 1. Due to the isometries (38) and (41), this implies

$$k! \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2 = \|I^{n,k}(f^{n,k})\|_{L^2(\Omega, \mathcal{F}, P)}^2 \rightarrow \|I^k(f^k)\|_{L^2(\Omega, \mathcal{F}, P)}^2 = k! \|f^k\|_{L^2([0,\infty)^k)}^2. \quad (48)$$

Moreover, for every $g \in \mathcal{E}$, we obtain

$$\begin{aligned} \langle g^{\otimes k}, \widehat{f^{n,k}} - f^k \rangle_{L^2([0,\infty)^k)} &= (SI^k(\widehat{f^{n,k}}))(g) - (SI^k(f^k))(g) \\ &= \left((SI^k(\widehat{f^{n,k}}))(g) - (S^n I^{n,k}(f^{n,k}))(\check{g}^n) \right) + \mathbb{E} \left[I^{n,k}(f^{n,k}) \exp^{\diamond n}(I^n(\check{g}^n)) - I^k(f^k) \exp^{\diamond}(I(g)) \right] \\ &\rightarrow 0, \end{aligned}$$

by Propositions 3 and 33, and the $L^2(\Omega, \mathcal{F}, P)$ -convergence of $(I^{n,k}(f^{n,k}))_{n \in \mathbb{N}}$ to $I^k(f^k)$. Since the set $\{g^{\otimes k}, g \in \mathcal{E}\}$ is total in $\widetilde{L^2}([0,\infty)^k)$, we may conclude that $(\widehat{f^{n,k}})$ converges weakly in $\widetilde{L^2}([0,\infty)^k)$ to f^k by [Yosida (1995), Theorem V.1.3]. Finally, (48) turns this weak convergence into strong $L^2([0,\infty)^k)$ -convergence. In particular, the k th coefficient in the chaos decomposition of the limiting random variable X is the strong $L^2([0,\infty)^k)$ -limit of $(\widehat{f^{n,k}})$, as asserted. It remains to show (40). However, by (47) and the isometries for (discrete) multiple Wiener integrals,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=m}^{\infty} k! \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2 &= \lim_{n \rightarrow \infty} \sum_{k=m}^{\infty} \|I^{n,k}(f^{n,k})\|_{L^2(\Omega, \mathcal{F}, P)}^2 \\ &= \lim_{n \rightarrow \infty} \left(\|\pi_{\mathcal{H}^n} X^n\|_{L^2(\Omega, \mathcal{F}, P)}^2 - \sum_{k=0}^{m-1} \|I^{n,k}(f^{n,k})\|_{L^2(\Omega, \mathcal{F}, P)}^2 \right) \\ &= \|X\|_{L^2(\Omega, \mathcal{F}, P)}^2 - \sum_{k=0}^{m-1} \|I^k(f^k)\|_{L^2(\Omega, \mathcal{F}, P)}^2 \rightarrow 0 \end{aligned}$$

as m tends to infinity.

‘(ii) \Rightarrow (i)’: In order to lighten the notation, we again denote the function $f_{X^n}^{n,k}$ from (43) by $f^{n,k}$. Assuming (ii), the strong $L^2([0,\infty)^k)$ -limit of $\widehat{f^{n,k}}$ exists and will be denoted f^k . We first show that $(I^{n,k}(f^{n,k}))$ converges to $I^k(f^k)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ for all $k \in \mathbb{N}_0$ by means of Theorem 1. To this end, we observe that, for every $g \in \mathcal{E}$,

$$\begin{aligned} (S^n I^{n,k}(f^{n,k}))(\check{g}^n) &= \left((S^n I^{n,k}(f^{n,k}))(\check{g}^n) - (S I^k(\widehat{f^{n,k}}))(g) \right) + (S I^k(\widehat{f^{n,k}}))(g) \\ &\rightarrow (S I^k(f^k))(g) \end{aligned}$$

by Proposition 33 and the isometry for continuous multiple Wiener integrals. Moreover, again, by the isometries for discrete and continuous multiple Wiener integrals,

$$\mathbb{E} \left[(I^{n,k}(f^{n,k}))^2 \right] = k! \|f^{n,k}\|_{L_n^2(\mathbb{N}^k)}^2 = k! \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2 \rightarrow k! \|f^k\|_{L^2([0,\infty)^k)}^2 = \mathbb{E} \left[(I^k(f^k))^2 \right].$$

So, Theorem 1 applies indeed. With the $L^2(\Omega, \mathcal{F}, P)$ -convergence of $I^{n,k}(f^{n,k})$ to $I^k(f^k)$ at hand, we can now decompose, for every $m \in \mathbb{N}$,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \mathbb{E} \left[\left| \pi_{\mathcal{H}^n} X^n - \sum_{k=0}^{\infty} I^k(f^k) \right|^2 \right] \\ &\leq 3 \limsup_{n \rightarrow \infty} \left(\left\| \sum_{k=0}^{m-1} I^k(f^k) - \sum_{k=0}^{m-1} I^{n,k}(f^{n,k}) \right\|_{L^2(\Omega, \mathcal{F}, P)}^2 + \sum_{k=m}^{\infty} \|I^k(f^k)\|_{L^2(\Omega, \mathcal{F}, P)}^2 \right. \\ &\quad \left. + \sum_{k=m}^{\infty} \|I^{n,k}(f^{n,k})\|_{L^2(\Omega, \mathcal{F}, P)}^2 \right) \end{aligned}$$

$$= 3 \sum_{k=m}^{\infty} k! \|f^k\|_{L^2([0,\infty)^k)}^2 + 3 \limsup_{n \rightarrow \infty} \sum_{k=m}^{\infty} k! \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2. \quad (49)$$

By Fatou's lemma,

$$\sum_{k=m}^{\infty} k! \|f^k\|_{L^2([0,\infty)^k)}^2 = \sum_{k=m}^{\infty} k! \lim_{n \rightarrow \infty} \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2 \leq \liminf_{n \rightarrow \infty} \sum_{k=m}^{\infty} k! \|\widehat{f^{n,k}}\|_{L^2([0,\infty)^k)}^2.$$

Hence, letting m tend to infinity in (49), we observe, thanks to (40), that $(\pi_{\mathcal{H}^n} X^n)$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$. \square

We close this section with an example.

Example 35. Fix $X \in L^2(\Omega, \mathcal{F}, P)$. Theorem 29 with $X^n = X$ for every $n \in \mathbb{N}$, implies that the chaos coefficients f_X^k , $k \in \mathbb{N}_0$, of X are given as the strong $L^2([0, \infty)^k)$ -limit of

$$\widehat{f^{n,k}}(u_1, \dots, u_k) := \frac{1}{k!} \mathbb{E} \left[X \left(\prod_{l=1}^k \frac{B_{[nu_l]}^n - B_{([nu_l]-1)}^n}{1/n} \right) \right] \mathbf{1}_{\{|\{[nu_1], \dots, [nu_k]\} \cap \mathbb{N}|=k\}}.$$

This formula can be further simplified when X is \mathcal{F}_T -measurable. Then, one can show, analogously to Example 5 (ii), that the sequence $(\pi_{\mathcal{H}_{[nT]}} X)$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$.

Applying Theorem 29 with the latter sequence, shows that the chaos coefficients f_X^k , $k \in \mathbb{N}_0$, are the strong $L^2([0, \infty)^k)$ -limit of

$$\widehat{f^{n,k}}(u_1, \dots, u_k) := \frac{1}{k!} \mathbb{E} \left[X \left(\prod_{l=1}^k \frac{B_{[nu_l]}^n - B_{([nu_l]-1)}^n}{1/n} \right) \right] \mathbf{1}_{\{|\{[nu_1], \dots, [nu_k]\} \cap \{1, \dots, [nT]\}|=k\}}.$$

In this case, for each fixed $n \in \mathbb{N}$, only finitely many of the functions $\widehat{f^{n,k}}$, $k \in \mathbb{N}_0$, are not constant zero, and these are simple functions with finitely many steps sizes only.

These two approximation formulas for the chaos coefficients of X are one way to give a rigorous meaning of the heuristic formula

$$f_X^k(u_1, \dots, u_k) = \frac{1}{k!} \mathbb{E} \left[X \left(\prod_{l=1}^k \dot{B}_{u_l} \right) \right],$$

where \dot{B} is white noise, which is called Wiener's intuitive recipe in [Cutland and Ng (1991)]. The latter paper provides another rigorous meaning to Wiener's recipe via nonstandard analysis, which is closely related to our approximation formulas in the special case of symmetric Bernoulli noise. The authors show that

$$f_X^k(\circ t_1, \dots, \circ t_k) = \frac{1}{k!} \circ \mathbb{E} \left[x(b) \left(\frac{\Delta b_{t_1}}{\Delta t} \dots \frac{\Delta b_{t_k}}{\Delta t} \right) \right], \quad t_l \in T = \{j\Delta t, 0 \leq j < N^2\},$$

where N is infinite, $\Delta t = 1/N$, $b_t(\omega) = \sqrt{\Delta t} \sum_{s \leq t} \omega(s)$, $t \in T$, $\omega \in \Omega := \{-1, 1\}^T$, which is equipped with the internal counting measure, $x(b)$ is a lifting of X , \mathbb{E} is the expectation operator with respect to the internal counting measure, and the circle denotes the standard part.

6. STRONG L^2 -APPROXIMATION OF THE SKOROKHOD INTEGRAL AND THE MALLIAVIN DERIVATIVE

In this section, we apply the Wiener chaos limit theorem (Theorem 29) in order to prove strong L^2 -approximation results for the Skorokhod integral and the Malliavin derivative. For the construction of the approximating sequences we compose the discrete Skorokhod integral and the discretized Malliavin derivative with the orthogonal projection on \mathcal{H}^n , i.e. on the subspace of random variables which admit a discrete chaos decomposition in terms of multiple integrals with respect to the discrete time noise $(\xi_i^n)_{i \in \mathbb{N}}$.

We first treat the Malliavin derivative and aim at proving the following result.

Theorem 36. Suppose $(X^n)_{n \in \mathbb{N}}$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$ to X and, for every $n \in \mathbb{N}$, $\pi_{\mathcal{H}^n} X^n \in \mathbb{D}_n^{1,2}$. Then the following are equivalent:

- (i) $\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=m}^{\infty} k k! \|\widehat{f_{X^n}^{n,k}}\|_{L^2([0, \infty)^k)}^2 = 0$ (with $\widehat{f_{X^n}^{n,k}}$ as defined in (39)).
- (ii) $X \in \mathbb{D}^{1,2}$ and the sequence $(D_{[n, \cdot]}^n(\pi_{\mathcal{H}^n} X^n))_{n \in \mathbb{N}}$ converges to DX strongly in $L^2(\Omega \times [0, \infty))$ as n tends to infinity.

Note first, that by continuity of D_i^n for a fixed time $i \in \mathbb{N}$, we get

$$\begin{aligned} D_i^n(\pi_{\mathcal{H}^n} X^n) &= \sum_{|A| < \infty} \mathbb{E}[X^n \Xi_A^n] D_i^n \Xi_A^n = \sqrt{n} \sum_{|A| < \infty; i \in A} \mathbb{E}[X^n \Xi_A^n] \Xi_{A \setminus \{i\}}^n \\ &= \sqrt{n} \sum_{|B| < \infty; i \notin B} \mathbb{E}[X^n \Xi_{B \cup \{i\}}^n] \Xi_B^n. \end{aligned}$$

By the relation (42)–(43) between Walsh decomposition and discrete chaos decomposition, this identity can be reformulated as

$$D_i^n(\pi_{\mathcal{H}^n} X^n) = \sum_{k=1}^{\infty} k I^{n,k-1}(f_{X^n}^{n,k}(\cdot, i)). \quad (50)$$

Hence, the isometry for discrete multiple Wiener integrals (41) implies

$$\frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} [|D_i^n(\pi_{\mathcal{H}^n} X^n)|^2] = \sum_{k=1}^{\infty} k k! \|\widehat{f_{X^n}^{n,k}}\|_{L^2([0, \infty)^k)}^2, \quad (51)$$

i.e.,

$$\pi_{\mathcal{H}^n} X^n \in \mathbb{D}_n^{1,2} \Leftrightarrow \sum_{k=1}^{\infty} k k! \|\widehat{f_{X^n}^{n,k}}\|_{L^2([0, \infty)^k)}^2 < \infty. \quad (52)$$

This is in line with the characterization of the continuous Malliavin derivative in terms of the chaos decomposition, see e.g. [Nualart (2006)], which we show to be equivalent to Definition 12 in the Appendix:

$$X \in \mathbb{D}^{1,2} \Leftrightarrow \sum_{k=1}^{\infty} k k! \|f_X^k\|_{L^2([0, \infty)^k)}^2 < \infty, \quad (53)$$

and, if this is the case,

$$D_t X = \sum_{k=1}^{\infty} k I^{n,k-1}(f_X^k(\cdot, t)), \text{ a.e. } t \geq 0, \quad \int_0^{\infty} \mathbb{E}[(D_t X)^2] dt = \sum_{k=1}^{\infty} k k! \|f_X^k\|_{L^2([0, \infty)^k)}^2. \quad (54)$$

After these considerations on the connection between (discretized) Malliavin derivative and (discrete) chaos decomposition, the proof of Theorem 36 turns out to be rather straightforward.

Proof of Theorem 36. By Theorem 29 (in conjunction with Remark 27), we observe that, for every $k \in \mathbb{N}_0$, $(\widehat{f_{X^n}^{n,k}})_{n \in \mathbb{N}}$ converges to f_X^k strongly in $L^2([0, \infty)^k)$. Hence, by (51), (53), and (54),

$$\begin{aligned} (i) &\Leftrightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} k k! \|\widehat{f_{X^n}^{n,k}}\|_{L^2([0, \infty)^k)}^2 = \sum_{k=1}^{\infty} k k! \|f_X^k\|_{L^2([0, \infty)^k)}^2 < \infty \\ &\Leftrightarrow X \in \mathbb{D}^{1,2} \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E} [|D_i^n(\pi_{\mathcal{H}^n} X^n)|^2] = \int_0^{\infty} \mathbb{E}[(D_t X)^2] dt. \end{aligned}$$

Hence, the asserted equivalence is a direct consequence of Theorem 13. \square

We now wish to derive an analogous strong approximation result for the Skorokhod integral, which requires some additional notation. For every $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $k \in \mathbb{N}_0$, we denote

$$\mathfrak{f}_{Z^n}^{n,k}(i_1, \dots, i_k, i) := f_{Z^n}^{n,k}(i_1, \dots, i_k) = \begin{cases} \mathbb{E} \left[\frac{n^{k/2}}{k!} Z_i^n \Xi_{\{i_1, \dots, i_k\}}^n \right], & |\{i_1, \dots, i_k\} \cap \mathbb{N}| = k \\ 0, & \text{otherwise.} \end{cases}$$

Then, with $\pi_{\mathcal{H}^n} Z^n := (\pi_{\mathcal{H}^n} Z_i^n)_{i \in \mathbb{N}}$,

$$\sum_{k=0}^{\infty} k! \|\mathfrak{f}_{Z^n}^{n,k}\|_{L_n^2(\mathbb{N}^{k+1})}^2 = \|\pi_{\mathcal{H}^n} Z^n\|_{L_n^2(\Omega \times \mathbb{N})}^2 < \infty,$$

but $\mathfrak{f}_{Z^n}^{n,k}$ is symmetric in the first k variables only and does not, in general, vanish on the diagonal. For a function F in k variables, we denote its symmetrization by

$$\tilde{F}(y_1, \dots, y_k) = \frac{1}{k!} \sum_{\pi} F(y_{\pi(1)}, \dots, y_{\pi(k)}),$$

where the sum runs over the group of permutations of $\{1, \dots, k\}$. With this notation, $\tilde{\mathfrak{f}}_{Z^n}^{n,k} \mathbf{1}_{\partial_{k+1}^c}$ is an element of $\tilde{L}_n^2(\mathbb{N}^{k+1})$.

We can now state:

Theorem 37. *Suppose that, for every $n \in \mathbb{N}$, $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $\pi_{\mathcal{H}^n} Z^n \in D(\delta^n)$. Moreover, assume that $(Z^n)_{n \in \mathbb{N}}$ converges to Z strongly in $L^2(\Omega \times [0, \infty))$. Then, the following assertions are equivalent:*

- (i) $\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=m}^{\infty} k! \|\tilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}\|_{L_n^2(\mathbb{N}^k)}^2 = 0$.
- (ii) $Z \in D(\delta)$ and $(\delta^n(\pi_{\mathcal{H}^n} Z^n))$ converges to $\delta(Z)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ as n tends to infinity.

As a preparation of the proof we note that, for every $M \in \mathbb{N}$,

$$\begin{aligned} & \sum_{i=1}^M \mathbb{E}[\pi_{\mathcal{H}^n} Z_i^n | \mathcal{F}_{M,-i}^n] \frac{\xi_i^n}{\sqrt{n}} = \sum_{i=1}^M \sum_{|A| < \infty} \mathbb{E}[Z_i^n \Xi_A^n] \mathbb{E}[\Xi_A^n | \mathcal{F}_{M,-i}^n] \frac{\xi_i^n}{\sqrt{n}} \\ &= n^{-1/2} \sum_{i=1}^M \sum_{A \subset \{1, \dots, M\}} \mathbf{1}_{\{i \notin A\}} \mathbb{E}[Z_i^n \Xi_A^n] \Xi_{A \cup \{i\}}^n = n^{-1/2} \sum_{k=1}^M \sum_{B \subset \{1, \dots, M\}, |B|=k} \sum_{i \in B} \mathbb{E}[Z_i^n \Xi_{B \setminus \{i\}}^n] \Xi_B^n \\ &= n^{-1/2} \sum_{k=1}^M k \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k, i_1 < \dots < i_k} \mathbf{1}_{[1,M]}^{\otimes k}(i_1, \dots, i_k) \frac{1}{k} \sum_{j=1}^k \mathbb{E}[Z_{i_j}^n \Xi_{\{i_1, \dots, i_k\} \setminus \{i_j\}}^n] \Xi_{\{i_1, \dots, i_k\}}^n \\ &= \sum_{k=1}^M I^{n,k}(\tilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{[1,M]}^{\otimes k} \mathbf{1}_{\partial_k^c}). \end{aligned}$$

Hence, by the isometry for discrete multiple Wiener integrals,

$$\pi_{\mathcal{H}^n} Z^n \in D(\delta^n) \Leftrightarrow \sum_{k=1}^{\infty} k! \|\tilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}\|_{L_n^2(\mathbb{N}^k)}^2 < \infty, \quad (55)$$

and, if this is the case,

$$\delta^n(\pi_{\mathcal{H}^n} Z^n) = \sum_{k=1}^{\infty} I^{n,k}(\tilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}), \quad (56)$$

i.e., $f_{\delta^n(\pi_{\mathcal{H}^n} Z^n)}^{n,0} = 0$ and, for every $k \in \mathbb{N}$,

$$f_{\delta^n(\pi_{\mathcal{H}^n} Z^n)}^{n,k} = \tilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}.$$

For the proof of Theorem 37, we also provide the following variant of Theorem 29, ‘(i) \Rightarrow (ii)’, for stochastic processes.

Proposition 38. *Suppose $Z^n \in L^2_n(\Omega \times \mathbb{N})$ for every $n \in \mathbb{N}$ and $(Z^n_{[n \cdot]})$ converges strongly in $L^2(\Omega \times [0, \infty))$ to Z as n tends to infinity. Define the functions $\mathfrak{f}_Z^k \in L^2([0, \infty)^{k+1})$ via $\mathfrak{f}_Z^k(t_1, \dots, t_{k+1}) := f_{Z_{t_{k+1}}}^k(t_1, \dots, t_k)$. Then, for every $k \in \mathbb{N}_0$, as n tends to infinity,*

$$\widehat{\mathfrak{f}_{Z^n}^{n,k}} \rightarrow \mathfrak{f}_Z^k$$

strongly in $L^2([0, \infty)^{k+1})$.

Proof. The proof largely follows the arguments in the proof of Theorem 29. We spell it out for sake of completeness. Let $g, h \in \mathcal{E}$. Then, by the isometry for (discrete) multiple Wiener integrals, Corollary 34, and (7),

$$\begin{aligned} & \left\langle \widehat{\mathfrak{f}_{Z^n}^{n,k}}, (\check{g}^n)^{\otimes k} \otimes \check{h}^n \right\rangle_{L^2([0, \infty)^{k+1})} = \frac{1}{n} \sum_{i=1}^{\infty} \left\langle f_{Z_i^n}^{n,k}, (\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c} \right\rangle_{L^2_n(\mathbb{N}^k)} \check{h}^n(i) \\ &= \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[(\pi_{\mathcal{H}^n} Z_i^n) I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \check{h}^n(i) = \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \check{h}^n(i) \\ &\rightarrow \int_0^{\infty} \mathbb{E}[Z_s I^k(g^{\otimes k})] h(s) ds = \left\langle \mathfrak{f}_Z^k, g^{\otimes k} \otimes h \right\rangle_{L^2([0, \infty)^{k+1})}. \end{aligned} \quad (57)$$

As

$$\sup_{n \in \mathbb{N}} \left\| \widehat{\mathfrak{f}_{Z^n}^{n,k}} \right\|_{L^2([0, \infty)^{k+1})}^2 = \sup_{n \in \mathbb{N}} \int_0^{\infty} \mathbb{E} \left[\left| I^{n,k}(f_{Z_{[ns]}^n}^{n,k}) \right|^2 \right] ds \leq \sup_{n \in \mathbb{N}} \left\| Z_{[n \cdot]}^n \right\|_{L^2(\Omega \times [0, \infty))}^2 < \infty, \quad (58)$$

$(\check{g}^n)^{\otimes k} \otimes \check{h}^n \rightarrow g^{\otimes k} \otimes h$ strongly in $L^2([0, \infty)^{k+1})$ by (7), and the set $\{g^{\otimes k} \otimes h : g, h \in \mathcal{E}\}$ is total in the closed subspace of functions in $L^2([0, \infty)^{k+1})$, which are symmetric in the first k variables, we conclude again that $\widehat{\mathfrak{f}_{Z^n}^{n,k}}$ converges weakly to \mathfrak{f}_Z^k in this subspace. Hence, it only remains to argue that

$$\left\| \widehat{\mathfrak{f}_{Z^n}^{n,k}} \right\|_{L^2([0, \infty)^{k+1})}^2 \rightarrow \left\| \mathfrak{f}_Z^k \right\|_{L^2([0, \infty)^{k+1})}^2, \quad n \rightarrow \infty.$$

As

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \check{h}^n(i) &= \frac{1}{n} \sum_{i=1}^{\infty} (S^n I^{n,k}(f_{Z_i^n}^{n,k}))(\check{g}^n) \check{h}^n(i), \\ \int_0^{\infty} \mathbb{E}[Z_s I^k(g^{\otimes k})] h(s) ds &= \int_0^{\infty} (S I^k(f_{Z_s}^k))(g) h(s) ds, \end{aligned}$$

we may derive from (57)–(58) and Theorem 11, that $I^{n,k}(f_{Z_{[ns]}^n}^{n,k})$ converges to $I^k(f_Z^k)$ weakly in $L^2(\Omega \times [0, \infty))$. Thus,

$$\begin{aligned} \left\| \widehat{\mathfrak{f}_{Z^n}^{n,k}} \right\|_{L^2([0, \infty)^{k+1})}^2 &= \int_0^{\infty} \mathbb{E} \left[I^{n,k}(f_{Z_{[ns]}^n}^{n,k}) Z_s \right] ds + \int_0^{\infty} \mathbb{E} \left[I^{n,k}(f_{Z_{[ns]}^n}^{n,k}) (Z_{[ns]}^n - Z_s) \right] ds \\ &\rightarrow \int_0^{\infty} \mathbb{E} \left[I^k(f_Z^k) Z_s \right] ds = \left\| \mathfrak{f}_Z^k \right\|_{L^2([0, \infty)^{k+1})}^2. \end{aligned}$$

□

Proof of Theorem 37. By the linearity of the embedding operator $\widehat{(\cdot)}$, Minkowski inequality, Proposition 38, and Lemma 32, we obtain, for every $k \in \mathbb{N}_0$,

$$\left\| \widehat{\mathfrak{f}_{Z^n}^{n,k} \mathbf{1}_{\partial_{k+1}^c}} - \widetilde{\mathfrak{f}}_Z^k \right\|_{L^2([0, \infty)^{k+1})} = \left\| \widetilde{\mathfrak{f}_{Z^n}^{n,k} \mathbf{1}_{\partial_{k+1}^c}} - \widetilde{\mathfrak{f}}_Z^k \right\|_{L^2([0, \infty)^{k+1})} \leq \left\| \widehat{\mathfrak{f}_{Z^n}^{n,k} \mathbf{1}_{\partial_{k+1}^c}} - \mathfrak{f}_Z^k \right\|_{L^2([0, \infty)^{k+1})} \rightarrow 0$$

as n tends to infinity. Thus, due to Theorem 29 and (56),

$$(i) \Leftrightarrow (\delta^n(\pi_{\mathcal{H}^n} Z^n))_{n \in \mathbb{N}} \text{ converges strongly in } L^2(\Omega, \mathcal{F}, P).$$

Now, the implication ‘(ii) \Rightarrow (i)’ is obvious, while the converse implication is a consequence of Theorem 9. \square

Remark 39. As a by-product of the proof of Theorem 37, we recover, thanks to Theorem 29, the well-known chaos decomposition of the Skorokhod integral as

$$\delta(Z) = \sum_{k=1}^{\infty} I^k(\tilde{f}_Z^{k-1}).$$

7. BINARY NOISE

In this section, we specialize to the case of binary noise, i.e., we suppose that, for some constant $b > 0$,

$$P(\{\xi = -1/b\}) = \frac{b^2}{b^2 + 1}, \quad P(\{\xi = b\}) = \frac{1}{b^2 + 1}.$$

We illustrate, that in this binary case, our approximation formulas for the Malliavin derivative and the Skorokhod integral give rise to a straightforward numerical implementation.

We recall first that Malliavin calculus on the Bernoulli space is well-studied, see, e.g. [Holden et al. (1992)], [Leitz-Martini (2000)], [Privault (2009)], and the references therein, usually with the aim to explain the main ideas of Malliavin calculus by discussing the analogous operators in the simple toy setting. Note first that $L^2(\Omega, \mathcal{F}_i^n, P)$ equals \mathcal{H}_i^n in the binary case (and in this case only) by observing that both spaces have dimension 2^i in this case. Hence, $L^2(\Omega, \mathcal{F}^n, P)$ coincides with \mathcal{H}^n for binary noise, and we can drop the orthogonal projections $\pi_{\mathcal{H}^n}$ on \mathcal{H}^n in the statement of all previous results. In particular, every random variable $X^n \in L^2(\Omega, \mathcal{F}^n, P)$ then admits a chaos decomposition in terms of discrete multiple Wiener integrals, and the representations of the discretized Malliavin derivative and the discrete Skorokhod integral in terms of the discrete chaos in Section 6 show that these operators coincide with the Malliavin derivative and the Skorokhod integral on the Bernoulli space, see [Privault (2009)].

In the binary case, the representations for the discrete Malliavin derivative and the discrete Skorokhod integral can be simplified considerably. Suppose $X^n \in L^2(\Omega, \mathcal{F}^n, P)$. Then, there is a measurable map $F_{X^n} : \mathbb{R}^\infty \rightarrow \mathbb{R}$ such that $X^n = F_{X^n}(\xi_1^n, \xi_2^n, \dots)$. A direct computation shows that, for every $i \in \mathbb{N}$,

$$\begin{aligned} D_i^n X &= \sqrt{n} \mathbb{E}[\xi_i^n F_{X^n}(\xi_1^n, \xi_2^n, \dots) | \mathcal{F}_{-i}^n] \\ &= \frac{\sqrt{nb}}{b^2 + 1} (F_{X^n}(\xi_1^n, \dots, \xi_{i-1}^n, b, \xi_{i+1}^n, \dots) - F_{X^n}(\xi_1^n, \dots, \xi_{i-1}^n, -1/b, \xi_{i+1}^n, \dots)), \end{aligned} \quad (59)$$

hence, the Malliavin derivative becomes a difference operator. Moreover, for $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $N \in \mathbb{N}$, the discrete Skorokhod integral can be rewritten as

$$\delta^n(Z^n \mathbf{1}_{[1, N]}) = \sum_{i=1}^N Z_i^n \frac{\xi_i^n}{\sqrt{n}} - \frac{1}{n} \sum_{i=1}^N (\xi_i^n)^2 D_i^n Z_i^n,$$

which can either be derived from [Privault (2009), Proposition 1.8.3] or by expanding Z_i^n in its Walsh decomposition and noting that, for every finite subset $A \subset \mathbb{N}$,

$$(\Xi_A^n - \mathbb{E}[\Xi_A^n | \mathcal{F}_{-i}^n]) \sqrt{n} \xi_i^n = \begin{cases} \sqrt{n} \Xi_{A \setminus \{i\}}^n (\xi_i^n)^2, & i \in A \\ 0, & i \notin A \end{cases} = (\xi_i^n)^2 D_i^n \Xi_A^n.$$

Hence, for $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $N \in \mathbb{N}$,

$$\delta^n(Z^n \mathbf{1}_{[1, N]}) = \sum_{i=1}^N F_{Z_i^n}(\xi_1^n, \xi_2^n, \dots) \frac{\xi_i^n}{\sqrt{n}}$$

$$- \frac{(\xi_i^n)^2 b}{\sqrt{n}(b^2 + 1)} (F_{Z_i^n}(\xi_1^n, \dots, \xi_{i-1}^n, b, \xi_{i+1}^n, \dots) - F_{Z_i^n}(\xi_1^n, \dots, \xi_{i-1}^n, -1/b, \xi_{i+1}^n, \dots)). \quad (60)$$

Recall that the discrete noise $(\xi_i^n)_{i \in \mathbb{N}}$, can be constructed from the underlying Brownian motion $(B_t)_{t \in [0, \infty)}$ via a Skorokhod embedding as

$$\xi_i^n = \sqrt{n} (B_{\tau_i^n} - B_{\tau_{i-1}^n}),$$

where, in the binary case,

$$\tau_0^n := 0, \quad \tau_i^n := \inf \left\{ s \geq \tau_{i-1}^n : B_s - B_{\tau_{i-1}^n} \in \left\{ \frac{b}{\sqrt{n}}, \frac{-1}{b\sqrt{n}} \right\} \right\}, \quad (61)$$

and the Brownian motion at the first-passage times τ_i^n can be simulated by the acceptance-rejection algorithm of [Burq and Jones (2008)].

We close this paper by a toy example which illustrates how to numerically compute Skorokhod integrals by our approximation results.

Example 40. *In this example, we approximate the Skorokhod integral $\delta(Z)$ for the process*

$$Z_t = \text{sign}(1/2 - t)(B_1 B_{1-t} - (1 - t)) \mathbf{1}_{[0,1]}(t), \quad t \geq 0,$$

where we choose the sign-function to be rightcontinuous at 0. For the discrete time approximation we consider

$$Z_i^n = \text{sign}(1/2 - i/n) (B_n^n B_{n-i}^n - (1 - i/n)) \mathbf{1}_{[1, n-1]}(i), \quad i \in \mathbb{N},$$

and note that $(Z_{\lfloor nt \rfloor}^n)$ converges to Z_t for almost every $t \geq 0$ in probability by (1). Hence, by uniform integrability and dominated convergence, it is easy to check that $(Z_{\lfloor n \cdot \rfloor}^n)_{n \in \mathbb{N}}$ converges to Z strongly in $L^2(\Omega \times [0, \infty))$. We next observe that in the discrete chaos decomposition of

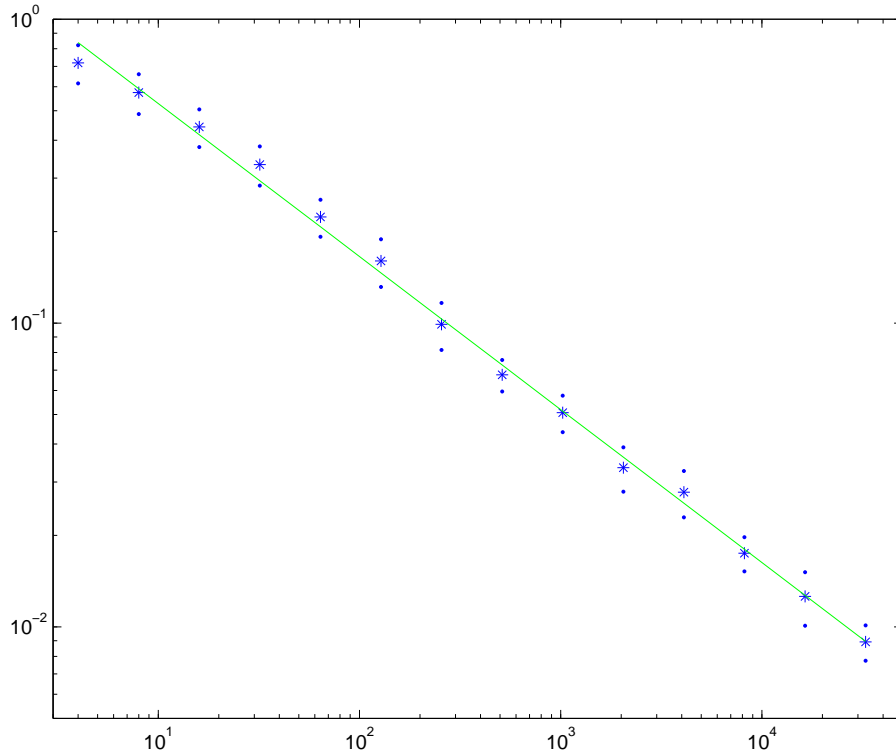


FIGURE 1. Log-log plot of the simulated strong $L^2(\Omega, \mathcal{F}, P)$ -approximation as the number of time steps increases.

$\delta^n(Z^n)$, all the coefficient functions $f_{\delta^n(Z^n)}^{n,k}$ for $k \geq 4$ vanish, because Z_i^n is a polynomial of degree 2 in B^n . Hence, the tail condition in Theorem 37 is trivially satisfied and, consequently, $(\delta^n(Z^n))_{n \in \mathbb{N}}$ converges to $\delta(Z)$ strongly in $L^2(\Omega, \mathcal{F}, P)$. We now suppose that B^n is constructed via the Skorokhod embedding (61) and simulate, for $n = 4, 8, \dots, 2^{15}$, 10000 independent copies $(B^{n,l})_{l=1, \dots, 10000}$ of B^n by the Burq&Jones algorithm. The corresponding realizations of $\delta^n(Z^n)$ and $\delta(Z)$ along the l th trajectory of the underlying Brownian motion are denoted $\delta_l^n(Z^n)$ and $\delta_l(Z)$, $l = 1, \dots, 10000$, respectively. For the discrete Skorokhod integral we implement formula (60) with $N = n$, while for the continuous Skorokhod integral we exploit that it can be computed analytically and equals

$$\delta(Z) = B_1 B_{1/2}^2 - \frac{B_1}{2} - B_{1/2}.$$

Figure 1 shows, in the case of symmetric binary noise ($b = 1$), a log-log-plot of the empirical mean (indicated by crosses) of $|\delta_l^n(Z^n) - \delta_l(Z)|^2$, $l = 1, \dots, 10000$, and the corresponding (asymptotical) 95%-confidence bounds (indicated by dots) as the number of time steps n increases. A linear regression (solid line) exhibits a slope of -0.5036 and, thus, indicates that strong $L^2(\Omega, \mathcal{F}, P)$ -convergence takes place at the expected rate of $1/2$.

APPENDIX A. S -TRANSFORM CHARACTERIZATION OF THE MALLIAVIN DERIVATIVE

In this appendix, we prove the equivalence between the definition of the Malliavin derivative in terms of the S -transform (Definition 12) and the more classical characterization in terms of the chaos decomposition, see (53)–(54).

Proposition 41. *Suppose $X = \sum_k I^k(f_X^k) \in L^2(\Omega, \mathcal{F}, P)$. Then, the following are equivalent:*

- (i) *There is a stochastic process $Z \in L^2(\Omega \times [0, \infty))$ such that for every $g, h \in \mathcal{E}$,*

$$\int_0^\infty (SZ_s)(g)h(s)ds = \mathbb{E} \left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right].$$

- (ii) $\sum_{k=1}^\infty k k! \|f_X^k\|_{L^2([0, \infty)^k)}^2 < \infty$.

If this is the case, then $Z_t = \sum_{k=1}^\infty k I^{n,k-1}(f_X^k(\cdot, t))$ for almost every $t \geq 0$.

Proof. We first note that, for every $f, g \in \mathcal{E}$,

$$\exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) = \sum_{k=1}^\infty \frac{1}{(k-1)!} I^k((g^{\otimes(k-1)} \otimes h)), \quad (62)$$

which can be verified by computing the S -transform of both sides. By the Cauchy-Schwarz inequality, we obtain for every $f, g \in \mathcal{E}$,

$$\begin{aligned} & \sum_{k=1}^\infty \int_{[0, \infty)^k} |k f_X^k(x) (g^{\otimes k-1} \otimes h)(x)| dx \\ & \leq \left(\sum_{k=1}^\infty k! \|f_X^k\|_{L^2([0, \infty)^k)}^2 \right)^{1/2} \left(\sum_{k=1}^\infty \frac{k}{(k-1)!} \|g\|_{L^2([0, \infty))}^{2(k-1)} \|h\|_{L^2([0, \infty))}^2 \right)^{1/2} < \infty. \end{aligned} \quad (63)$$

Hence, Fubini's theorem implies

$$\sum_{k=1}^\infty \int_{[0, \infty)^k} k f_X^k(x) (g^{\otimes k-1} \otimes h)(x) dx = \int_0^\infty \left(\sum_{k=1}^\infty \int_{[0, \infty)^{k-1}} k f_X^k(x, t) g^{\otimes k-1}(x) \right) h(t) dt,$$

i.e., by (62) and the isometry for multiple Wiener integrals,

$$\mathbb{E} \left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds \right) \right] = \int_0^\infty \left(\sum_{k=1}^\infty \int_{[0, \infty)^{k-1}} k f_X^k(x, t) g^{\otimes k-1}(x) \right) h(t) dt \quad (64)$$

for every $g, h \in \mathcal{E}$.

‘(i) \Rightarrow (ii)’: Assuming (i) and noting that (64) holds for every $g, h \in \mathcal{E}$, we observe that for every $g \in \mathcal{E}$, $\alpha \in \mathbb{R}$, and Lebesgue-almost every $s \in [0, \infty)$,

$$\sum_{k=1}^{\infty} \alpha^{k-1} \langle f_{Z_s}^{k-1}(\cdot), g^{\otimes(k-1)} \rangle_{L^2([0, \infty)^{k-1})} = (SZ_s)(\alpha g) = \sum_{k=1}^{\infty} \alpha^{k-1} \langle k f_X^k(\cdot, s), g^{\otimes(k-1)} \rangle_{L^2([0, \infty)^{k-1})}.$$

(Note, that the Lebesgue null set can be chosen independent of g , α . Indeed, one can first take $\alpha \in \mathbb{Q}$ and step functions g with rational step sizes and interval limits, and then pass to the limit). Comparing the coefficients in the power series and noting that $\{g^{\otimes k}, g \in \mathcal{E}\}$ is total in $\widetilde{L^2}([0, \infty)^k)$, we obtain, for every $k \geq 1$ and almost every $s \in [0, \infty)$,

$$k f_X^k(\cdot, s) = f_{Z_s}^{k-1}. \quad (65)$$

Therefore, the isometry for multiple Wiener-Itô integrals implies

$$\sum_{k=1}^{\infty} k k! \|f_X^k\|_{L^2([0, \infty)^k)}^2 = \int_0^{\infty} \mathbb{E}[|Z_s|^2] ds < \infty. \quad (66)$$

‘(ii) \Rightarrow (i)’: Define $Z_t = \sum_{k=1}^{\infty} k I^{n, k-1}(f_X^k(\cdot, t))$. Assuming (ii), we observe by the first identity in (66) that Z belongs to $L^2(\Omega \times [0, \infty))$. By the isometry for multiple Wiener integrals and the chaos decomposition of a Wick exponential we get, for every $g, h \in \mathcal{E}$.

$$\int_0^{\infty} (SZ_s)(g) h(s) ds = \int_0^{\infty} \left(\sum_{k=1}^{\infty} \int_{[0, \infty)^{k-1}} k f_X^k(x, t) g^{\otimes(k-1)}(x) dx \right) h(t) dt.$$

Hence, (64) concludes. \square

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SAARLAND UNIVERSITY, DEPARTMENT OF MATHEMATICS PO Box 151150, D-66041 SAARBRÜCKEN, GERMANY,
 UNIVERSITY OF MANNHEIM, INSTITUTE OF MATHEMATICS A5,6, D-68131 MANNHEIM, GERMANY.
E-mail address: bender@math.uni-sb.de, parczewski@math.uni-mannheim.de